[illegible] Mathematics Teacher

DEVOTED TO THE INTERESTS OF MATHEMATICS IN JUNIOR AND SENIOR HIGH SCHOOLS

Vo[illegible] VI | MARCH, 1923 | NUMBER 3

Published by the

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

CAMP HILL, PA.

[illegible] second-class matter [illegible], at the Post Office at [illegible] under the Act of [illegible]. Acceptance for mailing at [illegible] provided for in Section [illegible] Act of October 3, 1917, [illegible]

[illegible]THE[illegible]TICS TEACHER

[illegible] JOURN[illegible] NATIONAL COUNCIL OF TEACHERS [illegible]MATICS

[illegible]

[illegible], Editor[illegible] EUGENE R. SMITH, Associate Editor
[illegible] MARIE GUGLE
[illegible]AYLOR[illegible] JOHN W. [illegible]
[illegible], Business Manager

THE [illegible] COUNCIL OF TEACH[illegible] OF MATHEMATICS

[illegible] J. H. MINN[illegible] University of Pennsylvania, Philadelphia, Pa.
Vice Pres[illegible] MISS [illegible] WEEKS, St. Louis.
[illegible] E. A. FOBERG, Harrisburg, Pa.
[illegible] WE[illegible], Detroit, Mich.
[illegible] AUS[illegible], Oak Park, Ill.
MISS MAR[illegible], Columbus, Ohio.
[illegible] Co[illegible] MISS GERT[illegible] ALLEN, Oakland, Cal.
[illegible] HARRY WE[illegible]LER, Worcester, Mass.
[illegible] A. AUSTIN, Fresno, Cal.
[illegible] W. RANKIN, JR., Decatur, Ga.

[illegible]nization [illegible] ject the advancement of mathematics [illegible] junior and [illegible] schools.

[illegible] interes[illegible] matics and mathematics teaching are [illegible] membership[illegible] bers receive the official journal of [illegible] Coun[illegible] TEACHER,—which appears [illegible] August and September.

[illegible] should be addressed to

[illegible]
[illegible], Yonkers[illegible] School of Teachers College,
[illegible] New York City

[illegible] other business matters should

[illegible] A. FOB[illegible] Hill, Pa., and
State Dep[illegible] Instruction, Harrisburg, Pa.

[illegible]PTI[illegible] PER YEAR (eight numbers)

THE MATHEMATICS TEACHER

VOLUME XVI MARCH, 1923 NUMBER 3

THE THORNDIKE PHILOSOPHY OF TEACHING THE PROCESSES AND PRINCIPLES OF ARITHMETIC

MIDDLESEX A. BAILEY
N. Y. T. School for Teachers

The object of this article is to review the Thorndike system of teaching the processes and principles of arithmetic as explained in the "Psychology of Arithmetic" (P), "The New Methods of Arithmetic" (M), and "The Three-Book Series of Arithmetics (I, II, III), with the hope, by retaining the good and replacing the bad, of finding a better system.

THE THORNDIKE OUTLINE

Following is an outline of the system gathered from general statements and individual applications, and illustrated in the teaching of multiplication by a number of one order (I, 67). Mr. Thorndike says (I, vii); "The best way to secure eventual insight into the principles governing arithmetical operations is to learn to operate by imitation and the extension of past knowledge, then to make sure that the operation is right by verification from known facts, and last of all to learn why it is right and must be right."

I. Introduce each process and principle by a problem illustrating its need.

"The children of the third grade are to have a picnic. 32 are going. How many sandwiches will they need if each of the 32 children has 4 sandwiches?"

II. Tell the learner what to do.

"Here is a quick way to find out:—Think "4 × 2," write 8 under the 2 in ones' column. Think "4 × 3," write 12 under the 3 in tens' column."

$$\begin{array}{r} 32 \\ 4 \\ \hline 128 \end{array}$$

III. Require him to verify his answers from known facts.

"Prove the answer by adding four 32's."

IV. Expect him to conclude that the procedure is right "Because doing so always gives the right answer."

Satisfy yourself that the process of multiplication is right by solving the following problems by multiplication and by verifyng the answers by addition.

First Step. Mr. Thorndike says of this step (P, 266) "Dewey and others following him, have emphasized the desirability of having pupils do their work as active seekers, conscious of problems whose solution satifies some real need of their natures. Other things being equal, it is unwise, they argue for pupils to be led along blindfolded as it were by the teacher and textbook, not knowing where they are going or why they are going there. They ought rather to have some living purpose, and be zealous for its attainment".

For our "Proposed Outline", suppose we retain this step. The problem selected must illustrate the need of the *identical* procedure which is to be taught, must be true to *actual* conditions, and must conform as closely as possible to the *life* interests of the learner.

Mr. Thorndike is not always careful to observe these strictures. In the illustration, his problem calls for making 32 the multiplier instead of 4. The 5th problem in his series is, "One long trolley car holds 42 men. Four long cars hold . . . men." This meets the first requirement but violates the second. A long trolley car may have seats for 42 men but it will hold an indefinite number.

Second Step. We shall wish, I think, to replace this step by something better.

The learner should be an earnest seeker, working with the teacher to discover new procedures. He can not be such if he starts in by using an unknown method at dictation. During the period that he is following directions and before he verifies the answer, he is in total darkness; he is proceeding blindly. The true step, as I think, is for the learner to solve the problem by some process known to him, and if possible, without the aid of the teacher. By a later step, under the teacher's guidance, he may pass to the shorter process.

Third Step. The process of verification as employed by Mr. Thorndike is sometimes illogical, although it is not so in the previous illustration.

Take his verification of the procedure of multiplication by a number of two orders (I,149) and (I,132).

213	Think "2 3's are 6," write the 6 under the 2	42
42	of 42 in the ones' column	213
——		——
426	Think "4 3's are 12," write the 2 under the 4	126
852	of 42 in the tens" column.	42
——		84
8946		——
	Proceed in the same way with 42 multiplied by 213.	8946

The argument runs: Since the answer found by the second procedure is the same as that found by the first, the first process must be right. It is false and begs the question. The method of procedure may be wrong and yet the answers may agree. To show this, let us prove by the same kind of reasoning that the product is 8847. Instead of beginning at the right, we will begin at the left; instead of carrying to the left, we will carry to the right.

213	Think "4 2's are 8," write 8 under 4, the figure	42
42	of the multiplier that produces it. Think "4 1's	213
——	are 4," write 4. Think "4 3's are 12," write	——
8421	2 and carry 1.	84
426	Think "2 2's are 4," write 4 under 2, the figure	42
——	of the multiplier that produces it.	27
8847		——
	Proceed in the same way with 42 multiplied by 213.	8847

Our verification that 213 multiplied by 42 is 8847 is as true as Mr. Thorndike's that the product is 8946. The difference is that his method happens to be right while ours is wrong; we may not carry to the right, but the verification does not disclose the error. He also begs the question because, to prove his method he uses the very procedure which he is trying to establish. The child will not question this reasoning, but Mr. Thorndike is not justified in disregarding the laws of thought.

Even if we check by dividing the product by one of the factors, comparing the result with the other factor, we can not get out of the fatal logical circle because we still carry to the right.

42	Think what must be multiplied by 2 to make 8,
213) 8847	write 4.
8421	Think "4 2's are 8," write 8. Think "4 1's are 4,"
——	write 4. Think "4 3's are 12," write the 2 and carry
426	1. . . .
426	
——	

We must not conclude that checking multiplication by interchanging the terms is not logical and valuable in its place, but that place is after the validity of the process has been established. It may be used to correct errors in the execution of the process, but not to prove the process itself.

Take his verification for the procedure of placing the decimal point in the multiplication of decimals (II,82) and (M,42).

$$\begin{array}{r} 2.45 \\ \underline{1.6} \\ 1470 \\ \underline{245} \\ 3.920 \end{array} \qquad\qquad \begin{array}{r} .039 \\ \underline{.039} \\ 351 \\ \underline{117} \\ .001521 \end{array}$$

How can you be sure that 1.6×2.45 is not 39.2? Ans. Because the result can not be as great as 2×3 or 6 units.

How can you be sure that the product is not .392? Ans. Because the result can not be as small as 2×1 or 2 units.

How can you be sure that $.039 \times .039$ is not .01521? Ans. Because the result can not be as great as $.04 \times .04$ or .0016.

How can you be sure that the product is not .0001521? Ans. Because the result can not be as small as $.03 \times .03$ or .0009.

In the case of 1.6×2.45, which, by the way, is an example in mixed decimals and not in pure decimals, the verification presents nothing false, but in the case of $.039 \times .039$, it commits the logical error of "begging the question". That $.04 \times .04 = .0016$ is found by applying the rule whose verification is sought, viz., "Point off as many decimal places in the product as there are in the multiplicand and the multiplier together".

Here, again, the proposed check is valuable as a means of detecting errors in the application of the rule, but it can not be validly used as a means of proving the principle.

Verification when legitimately applied, as Mr. Thorndike applies it in many cases, is undoubtedly of great value. He says, (M,42), "The pupil verifies the procedure taught him for multiplying 412 by 3 by adding 412, 412, 412. He verifies the procedure taught him for dividing 675 by 25 by multiplying 27 by 25. He verifies the rule for adding fractions by objective measurement. He verifies the rule that the number of decimal places in the product equals the sum of the decimal places in the multiplier and in the multiplicand by comparing the results when the numbers are expressed as common fractions, checking $.25 \times .5$ by $\frac{1}{4} \times \frac{1}{2}$ and the like".

These cases are of two kinds, those in which verification consists in "Solving the problem by a process known to the learn-

er", comprising all of the above except two, and those in which verification, as in the case of proving division by multiplication, consists in checking the mechanical execution of the process.

In our "Proposed Outline" let us place the former, which should be the preparation for passing to a more convenient process, in our Second Step and the latter, which should be a proof that there has been no error in the work, in our Fifth Step.

Mr. Thorndike differs from other writers who favor inductive developments. They take several instances in which the phenomenon occurs and seek a common circumstance. Mr. Thorndike, alone, states the common circumstance in advance and asks the learner to discover that it is found in other instances in which the phenomenon occurs. Thus (M,43), "Work $6 \div \frac{3}{4}$ by $6 \times \frac{4}{3}$ and verify the result by dividing a 6-in. strip into $\frac{3}{4}$-in. lengths, work $2\frac{1}{2} \div \frac{5}{8}$ by $\frac{5}{2} \times \frac{8}{5}$ and verify the result by dividing a $2\frac{1}{2}$-in. strip into $\frac{5}{8}$-in. lengths, and similarly for other cases". Other writers reverse the process. Mr. Thorndike seems to "Put the cart before the horse".

The race could not have come to its present state of knowledge by following Mr. Thorndike's order. It could not have found the number of individuals in several groups by addition first, verifying by counting; it could not have found the number of individuals in several equal groups by multiplication first, verifying by adding. It must have reversed this order; it summarized for a time by counting, then for a time by adding, and then for a time by multiplying; always modifying the old to get the new.

Mr. Thorndike's order is unpsychological, unpedagogical and illogical.

Fourth Step. Mr. Thorndike says (M,42), "There are two sorts of reasons that may be given. The first sort refers back by a chain of arguments to axioms and the general nature of our arithmetical system. The second sort is very different, being in essence, "Because I find that doing so always gives the right answer."

The reason, "Because I find doing so always gives the right answer" is incorrectly stated. Without derivation from antecedent principles, it can not be *known* that a process will give

the right answer in any cases except those tried. Thus, in the foregoing illustrations where a strip is divided experimentally into $\frac{3}{4}$-in. lengths, $\frac{5}{8}$-in. lengths, and so on, to find the number of lengths, it can not be known that the answer in all cases will be the same as that found by inverting the divisor and multiplying because no reason appears why it should be so. The process is purely experimental; for aught we know it may fail in the very next trial. The fact that the principle can be validly established by other methods does not warrant a false conclusion from the premises used.

The reason in correct form, "Because I find that doing so gives the right answer in all cases tried", is too weak to satisfy the intelligence. If it is true, as Mr. Thorndike contends, that the child can not understand a stronger reason, we must be content with the weak. May it not be true however, that the child can understand reasons derived from antecedent principles if they are properly presented? I think that he can. My proposition, then, is to replace this reason by a stronger, a reason that shall appeal to the intelligence of the learner, and to place it as the Third Step of our outline, to be used by the learner in seeking for a shorter process.

THE PROPOSED OUTLINE

Let us state our "Proposed Outline" and illustrate it in the case of multiplication by a number of one order.

I. Introduce each process and principle by a problem illustrating its need.

A long trolley car has 32 seats. How many seats has 4 such cars?

II. Ask the learner to solve the problem by a method known to him.

Find the number of seats. Ans. The pupil solves by addition.

III. Invite him to join with you in finding a shorter process.

32	Let us try to find a shorter way.	
32	When you said "2, 4, 6, 8," you found the sum of	
32	4 . . .'s; you may say "4 2's are . . .," write 8	32
32	under . . . in . . . column.	4
——	You may then say, "4 . . .'s are . . .," write	——
128	. . . in . . . column.	128

Instead of writing 32 4 times, you may express the same thing more briefly by writing 32 . . . and placing . . . beneath.

IV. Require him to fix the new process in mind by solving one or more other problems, first by the old method and then by the new.

How many seats has 3 such cars? Solve by addition and then by multiplication.

V. Require the solution of other problems by the new method without statement of the rule, or by the statement of the rule and its application.

In either case, require a proof of the answer by some check.

Solve the following problems by multiplication and prove the answers by going over the work mentally.

The reason why Step Five offers a choice of two procedures is because, as shown in logic, it is possible to reason from a single instance to a single instance directly, in which case no rule is needed, or indirectly by passing to the general and then back again to the individual, in which case a rule is required. For the process of multiplication by a number of one order it is probably best, on account of its simplicity, to pass from one instance to another without a rule, while for the process of multiplication of decimals it is probably best on account of its complexity, to use the rule.

By Mr. Thorndike's development, the child is in the dark until after he has verified his answers; at the end, he understands that the process is right in the cases he has tried. By the proposed development, the child is in the light from the beginning; at the end, he understands that the new process is right always. Let us compare the two methods still further:

Multiplication by a Decimal

Thorndike Method	*Proposed Method*
1. Problem. Find the area of a rectangle whose length is .23 in. and whose width is .7 in.	1. Problem. Find the area of a rectangle whose length is .23 in. and whose width is .7 in.

2. *Apply the rule, "Multiply with decimals just as with whole numbers. Then point off as many decimal places in the product as there are in the multiplier and multiplicand together."*

$.23 \times .7 = .161$

3. Check the placing of the decimal point by giving to each digit its place value as the multiplication is performed (III. 13).

$$\begin{array}{r} .23 \\ \underline{.7} \\ .7 \times .03 = .021 \\ .7 \times .2 \;\; = .14 \\ \hline .7 \times .23 = .161 \end{array}$$

Note. This check begs the question unless the place values of the digits are expressed as common fractions. Thus:

$7/10 \times 3/100 = .021$

4. The rule is probably true, "Because doing so gives the right answer in all the cases I have tried."

Test it further, by solving the following examples by rule and by checking the answers as above.

5. Solve the following problems by rule.

2. Solve the problem.

Mary gets .161 sq. in. Here is her work:

$$.23 \times .7 = \frac{23}{100} \times \frac{7}{10} = \frac{161}{1000} = .161$$

She reduces the decimals to . . . , multiplies the . . . . for a new . . . . and the . . . . for a new . . . . , and reduces the resulting fraction to a . . . .

3. Let us examine her work for a shorter way.

$$\begin{array}{r} .23 \\ \underline{.7} \\ .161 \end{array}$$

The 161 of the answer is the product of the decimals regarded as . . . .

The number of decimal places in the answer is the number of decimal places in the . . . and the . . . . together.

Hence, for a shorter way, we may arrange the work . . . . and then:

Multiply as in integers and point off as many decimal places in the product as there are decimal places in the multiplicand and the multiplier together.

4. Multiply .039 by .27, first by reducing the decimals to common fractions and then by rule.

Note. It is best to require the pupil to state the rule exactly as it appears in the textbook. Otherwise, he will probably state it inaccurately.

5. Solve the following problems by rule and prove the multiplication by excess of nines.

By the "Thorndike Method", the pupil makes no attempt to solve the problem for himself, but follows the directions of his teacher. By the "Proposed Method", he solves the problem by himself and, under the guidance of his teacher finds a better method. By the one he is passive; by the other he is active. By the one he is trained; by the other he is educated.

Division by a Fraction

Thorndike Method	*Proposed Method*
1. Problem. "How many $\frac{3}{4}$-in. lengths are there in $2\frac{1}{4}$-in.?	1. Problem. How many $\frac{3}{4}$-in. lengths are there in $2\frac{1}{4}$-in.?
2. Apply the rule, "To divide a fraction multiply by its reciprocal." $\frac{9}{4} \times \frac{4}{3} = 3$	2. Solve the problem. The no. of $\frac{1}{4}$-in. lengths is 4 times $2\frac{1}{4}$ or 9; the no. of $\frac{3}{4}$ in. lengths is $\frac{1}{3}$ of 9 or 3. 1st Proof. $3 \times \frac{3}{4}$ in. is $2\frac{1}{4}$ in. 2d Proof. Measuring by the foot rule, we get 3 lengths.
3. Check the answer by measuring with your foot rule. The number of $\frac{3}{4}$-in. lengths is found to be 3.	3. Let us modify the above method. The no. of $\frac{3}{4}$-in. lengths is $\frac{9}{4} \div \frac{3}{4}$. We see from No. 2 that this division is performed by multiplying by 4 and taking $\frac{1}{3}$ of the result or by multiplying by $\frac{4}{3}$. Thus: $\frac{9}{4} \times \frac{4}{3} = 3$ ***To divide by a fraction multiply by its reciprocal.***
4. The rule is probably true, "Because it gives the right answer in all the cases I have tried." Test it further by solving the following problems by rule and checking the answers by your foot measure.	4. Let us solve a different type of problem. Mary says "$\frac{3}{4}$ of my money is 18c. How much have I? $\frac{1}{4}$ of her money is $\frac{1}{3}$ of 18c or 6c; $\frac{4}{4}$ of it is 4×6c or 24c. Proof. $\frac{3}{4}$ of 24c is 18c. Her money is 18c $\div \frac{3}{4}$. We see

that this division is performed by multiplying by $\frac{1}{3}$ of 4 or $\frac{4}{3}$ or the reciprocal of $\frac{3}{4}$.

$$18 \times \frac{4}{3} = 24$$

5. Solve the following problems by rule.

"How many fields of $2\frac{1}{4}$ A. each will 18 A. make

"What is the cost for one pound of sugar when you get $3\frac{1}{2}$ pounds for 25c?"

5. Solve the following problems by rule and check by the principle. "Multiply the divisor and the quotient; the result should be the dividend."

How many fields of $2\frac{1}{4}$ A. each will 18 A. make?

Ans. 8. Proof. $8 \times 2\frac{1}{4}$ A. = 18 A.

The introductory and practice problems given above illustrate division by a fraction, but may be solved more easily and more quickly without the reciprocal process. For this reason, they are not the best that could be selected. I employed them in the "Proposed Method" because they were the ones used by Mr. Thorndike.

Let us solve the last two in Mr. Thorndike's series and a fruit stand problem, both by rule and by analysis.

How many fields of $2\frac{1}{4}$ A. each will 18 A. make?

The no. of $2\frac{1}{4}$ A. in 18 A. is $18 \div 2\frac{1}{4}$ or $18 \div \frac{9}{4}$ or $18 \times \frac{4}{9}$ or 8.

The no. of $2\frac{1}{4}$ A. in 18 A. is the no. of 9 A. in 72 A. or 8.

What is the cost for one pound of sugar when you get $3\frac{1}{2}$ pounds for 25c?

1 pound costs $25c \div 3\frac{1}{2}$ or $25c \div \frac{7}{2}$ or $25c \times \frac{2}{7}$ or $7\frac{1}{7}$c.

If $3\frac{1}{2}$ pounds cost 25c, 7 pounds cost 50c; 1 pound costs $7\frac{1}{7}$c.

At 3 for 5c how many apples can be bought for 20c?

The no. of apples for 20c is $20 \div \frac{5}{3}$ or $20 \times \frac{3}{5}$ or 12.

At 5c each the no. of apples for 20c is 4; at 3 for 5c the no. is 3×4 or 12.

It is evident that, for the solution of such problems, the rule is both unnecessary and inconvenient. I am satisfied that it is seldom used by those who understand analysis. At all events, solutions by analysis should not be omitted, and should come before solutions by rule.

Problems that illustrate convenience and saving of time by the reciprocal rule arise in the use of formulae. Take, *e.g.*, the for-

mula for the area of a circle, $A = \pi r^2$. "If the area of a circle is 154 sq. ft., what is the radius?" $r^2 = 154 \div 3\frac{1}{7}$ or $154 \times \frac{7}{22}$ or 49; $r = 7$. *Ans.* 7 ft.

These considerations lead me to raise the question whether the reciprocal rule for division by a fraction may not be omitted until after the 6B grade?

To introduce this reciprocal forces the teaching of an unnecessary process and then the giving of unnecessary problems for its application. Some of our arithmetics develop this rule and illustrate its use by pages of examples like $\frac{47}{23} \div \frac{9}{46}$ and $83\frac{2}{3} \div \frac{117}{251}$. This seems foolish. The process is analogous to that of some algebras of spending a fourth of the book in the development of factoring and then another fourth in the reduction of equations made up to afford practice in that work—a task worthy of Sisyphus.

We will all agree, I think, that the solution by analysis, of problems involving fractional relations, is exceedingly valuable not only for practical purposes, but also for mental discipline. After such practice the passing to solutions by the reciprocal rule as suggested in the "Proposed Method" becomes simple.

Mr. Thorndike says (P. 154), "Other things being equal, reserve all explanations of why a process must be right until the pupils can use the process accurately, and have verified the fact that it is right. Except for the very gifted pupils, the ordinary preliminary deductive explanations of what must be done are probably useless as means of teaching the pupils what to do. I am not sure that the deductive proofs of why we place the decimal point as we do in the division by a decimal, or invert and multiply in dividing by a fraction are worth teaching at all."

The valid objections to the "Ordinary preliminary deductive explanations" are that the child has no part in them and that they are beyond his comprehension. The "Proposed Method" removes both of these objections. The valid objection to the Thorndike system is that the formation of bonds for solving the practical problems of life is taught through imitation and verification rather than through initiative and rationalization. The "Proposed Method" removes this objection.

By the suppression of reasoning from relations of terms, Mr. Thorndike's philosophy, instead of raising the efficiency of the

schools, lowers it. It encourages the tendency among both elementary and high school teachers to minimize intellectual activity on the part of pupils; to teach solutions of type problems through endless repetitions with little reference to underlying principles. At present, pupils on leaving school are almost helpless to think out by themselves the solution of problems that depart from type, or even to solve type problems when stated in unusual forms.

Let me illustrate by tests on reasoning given to candidates admitted to one of the training schools last September. Bear in mind that these candidates were all high school graduates from approved four-year courses.

"If A requires 2 days for a job and B 3 days how long will both together require?" Of 250 who answered the question, only 6, or about $2\frac{1}{2}\%$, obtained the right result. One answer was zero days, a few answers were 5 days, and others ranged between these two extremes. The popular answer was $2\frac{1}{2}$ days, obtained as follows: $2 + 3 = 5$; $\frac{1}{2}$ of $5 = 2\frac{1}{2}$. In a similar test given to candidates admitted in February of the preceding term, about 60% solved the problem, "Find to two decimal places what decimal part 57.5 is of 67.2," by dividing 67.2 by 57.5; they had the choice of one of two guesses and the majority guessed wrong —there could have been no reasoning.

The pupil feeds on husks who never gets reasoning in arithmetic of a type higher than, "Because I find that doing so gives the right answer in all the cases tried". In geography, we try to have the pupil give reasons for the direction of the trade winds, the precipitation of rain, etc.; in every branch of learning, even in the science of teaching, we seek reasons other than "Because doing so gives good results". Shall mathematics, alone, which above all sciences presents opportunities for reasoning from antecedent principles, be taught during the first six school years with no reason higher than, "Because I find that doing so gives the right answer in all the cases tried?"

PROBABILITY APPLIED TO GRADES

By E. J. MOULTON
Northwestern University, Evanston, Ill.

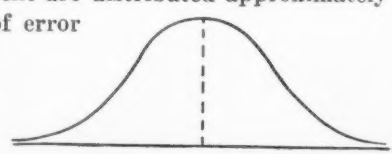
Measurements are subject to error, whether measurements of lengths or of intellectual attainments. A stick is measured, its length found to be 25.7 inches. Its true length, as determined from the average of a very great many careful measurements is likely to be somewhat different, perhaps 25.832 inches; the first measure is in error by .132. A grade is assigned to a student's performance—an oral answer, a theme, a test paper, or a semester's work—the grade being 80. The true grade, as determined by taking the average of grades on repeated assignments for the same performance is likely to be different, perhaps 81.4; the first grade is in error by 1.4.

Recognizing the existence of these errors, how should measurements be recorded? It is clear that there is little chance ordinarily of a measurement of the stick being in error by as much as 10 inches, but very likely a good chance that it is in error by as much as 0.01 in. Likewise, there is little chance that a grade is in error by as much as 30 but very likely a good chance that it is in error by as much as 2. It is clearly a matter of probability one way or another; the answer must be arrived at through a consideration of the probability of occurrence of errors of various magnitudes. In discussing the question we must fall back on the mathematics of probability.

Before attempting to answer the question I shall therefore state a few well-known theorems of probability, and apply them to the solution of three significant problems. If one accepts the solutions of these problems, the remainder of the paper can be read without encountering serious mathematical difficulties. Conclusions concerning grades are summarized at the end of the paper.

Theorems of Probability. Suppose errors of observation or judgment involving measurement are distributed approximately according to the normal law of error

$$(1) \quad \phi(E) = \frac{h}{\sqrt{\pi}} e^{-h^2E^2},$$



where E is the error and h a constant, the *measure of precision*. If h is small the errors are widely spread, if h is large the errors are concentrated near zero. One-half of the errors lie between $-r$ and r where

$$(2) \qquad r = .4769/h,$$

and one-half lie outside of those limits; r is called the *probable error* of a measurement.

Suppose a quantity F is determined from observations of F_1, F_2, . . ., and use of a linear relation of the form

$$(3) \qquad F = a_1F_1 + a_2F_2 + \dots$$

Let h, h_1, h_2, . . . be the measures of precision of F, F_1, F_2, ... Then it is a fundamental theorem that

$$(4) \qquad 1/h^2 = a_1^2/h_1^2 + a_2^2/h_2^2 + \dots$$

And if r, r_1, r_2, . . . are the corresponding probable errors, then

$$(5) \qquad r^2 = a_1^2r_1^2 + a_2^2r_2^2 + \dots .$$

The probability that an error lies between $-t$ and t is

$$(6) \qquad \Psi(t) = \frac{2h}{\sqrt{\pi}}\int_0^t e^{-h^2E^2}\,dE.$$

Let

$$(7) \qquad x = ht,\ y = hE.$$

Then

$$(8) \qquad \Psi(t) = \phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}\,dy.$$

This *error* (or *probability*) *integral*, $\phi(x)$, is tabulated in many books (see, for example, Jahnke-Emde, *Funktionentafeln*, or Peirce, *A Short Table of Integrals*, for values of x from 0.00 to 3.00). For values of $x > 3$, we use the approximation

$$(9) \qquad \phi(x) = 1 - \frac{e^{-x^2}}{\sqrt{\pi}x}\left(1 - \frac{1}{2x^2}\right)$$

which is correct to within $\dfrac{e^{-x^2}}{4\sqrt{\pi}x^4}$ which is less than 0.0000002.

Suppose a measurement is based upon the arithmetical mean of n measurements, each having the same probable error r_1. Then we have

$$a_1 = a_2 = \dots = 1/n,$$

and it follows from (5) that *the probable error, r, of the mean* is

$$r = \frac{r_1}{\sqrt{n}}; \tag{10}$$

it *varies universely as the square root of the number of measurements.*

Three Problems. I. Suppose a measurement is X_1, the true value being X and the probable error r_1. What is the probability P_1 that the true value is at least mr_1 greater than the actual measurement?

The probability that the error in measurement lies between $-mr_1$ and mr_1 is $\psi(mr_1)$; hence the required probability P_1 is $\frac{1}{2}[1-\psi(mr_1)]$. The substitution (7) in the form.

$$x = h_1 mr_1, \text{ where } h_1 = .4769/r_1,$$

gives the formula

$$P_1 = \tfrac{1}{2}[1-\phi(.477m)]. \tag{11}$$

A table of corresponding values of m and P_1 reads thus:

m	0	1	2	3	4	5	10
P_1	½	¼	1/11	1/45	1/280	1/2500	$1/2\text{x}10^{47}$.

II. Suppose measurements of two magnitudes, X_1 and Y_1 are made, the probable error of each being r_1 and the true values being X and Y. If Y_1 exceeds X_1 by mr_1, what is the probability P_2 that Y is less than X, that is, that the wrong one was measured to be the larger?

The measured difference is

$$\Delta_1 = Y_1 - X_1 = mr_1;$$

the probable error of this measurement is, by (5), $\sqrt{2}\,r_1$. Hence the probability that the error in the difference is between $-mr_1$ and mr_1 is $\Psi(mr_1)$ where $h = .4769/\sqrt{2}r_1$.

The required probability P_2, is therefore $\frac{1}{2}[1-\Psi(mr_1)]$. The substitution $x = hmr_1 = .337m$ gives the formula

$$P_2 = \tfrac{1}{2}[1-\phi(.337\,m)].$$

A table of corresponding values reads thus:

m	0	1	2	3	4	5	6	7
P_2	½	1/3	1/6	1/13	1/36	1/110	1/480	1/2400

m	8	9	10
P_2	1/14,000	1/110,000	1/1,000,000.

III. Suppose two measurements of each of two magnitudes are made, X_1 and X_2 for the first, Y_1 and Y_2 for the second, the probabe error of each being r_1. If Y_1 exceeds X_1 by mr_1, what is the probability P_3 that Y_2 is less than X_2, that is, that on repeated measurement the order of measured magnitudes is reversed?

Let

$$\Delta_1 = Y_1 - X_1, \qquad \Delta_2 = Y_2 - X_2;$$

the probable error of each of Δ_1 and Δ_2 is $\sqrt{2}\, r_1$, by (5), and hence the probable error of their difference $\Delta_1 - \Delta_2$ is $2\, r_1$. The probability that this difference lies between $-mr_1$, and mr_1 is $\psi\ (mr_1)$ where $h = .4769/2r_1$. Hence the probability, P_3, that this difference is greater than mr_1 is $\frac{1}{2}\ [1 - \Psi\ (mr_1)]$.

The substitution (7) in the form $x = hmr_1 = .238m$ gives

$$P_3 = \tfrac{1}{2}\ [1 - \phi\ (.238m)].$$

Corresponding values of m and P_3 are:

m	0	1	2	3	4	5	6
P_3	1/2	2/5	1/4	1/6	1/11	1/22	1/45
m	7	8	9	10			
P_3	1/110	1/280	1/800	1/2500			

Probable Error of a Grade. It has been found that in assigning grades an instructor has a pretty definite standard of measurement, and deviations of his grades from the standard occur approximately in accord with the normal law of error. The probable error of a grade depends upon the number of measurements upon which it is based, according to formula (10). If grades are calculated by taking the arithmetical mean of n independent grades, the probable error is inversely proportional to the square root of n.

Starch concluded (Science, 1913, vol. 38,: pp. 630-636) that the probable error of an instructor's grades, judged by the instructor's own standards, for a paper containing about ten answers, is about 1.75, grades being on the scale 0, 1, 2, . . ., 100. By an independent study I have found that for papers of "passing" grade the probable error for English is about 1.7, for astronomy about 1.9, and for algebra about 1.0. We will be near the truth for most cases if we assume as a basis for our discussion, as we shall, that the probable error is 2.0. The actual accur-

acy of an instructor's grades is greater in all cases tested experimentally, as far as I know. Our conclusions may therefore fairly be assumed to be conservative.

Grades on Individual Answers. It is fairly common to assign grades for individual answers on the scale, 0, 1, 2, . . ., 10. The probable error of a grade for a single answer is, from (10), $\sqrt{10}$ times as great as for the average of grades on ten answers,—all grades being on the same scale. Hence on the scale 0, 1, 2, . . ., 100 it is $2\sqrt{10} = 6.3$; on the scale 0, 1, 2, . . ., 10 it is 0.63.

If a grade of 7 is assigned to an answer, what is the probability that the true grade according to the instructor's standard is as high as 8? We have $r_1 = 0.63$, $mr_1 = 1.0$, $m = 1.6$, hence $P_1 = 1/7$; the probability is only one in seven. [If we take $r_1 = 1.0$, we find that the probability is one in four.]

If grades are assigned on the scale 0, 1, 2, 3, 4, 5, the system is equivalent to the scale 0, 2, 4, 6, 8, 10. If a grade of 6 on the latter scale is assigned what is the probability that the true grade should be as high as the next higher grade, 8? With $r_1 = .63$, $mr_1 - 2.0$, $m = 3.2$, we find $P_1 = 1/60$ (about); the probability is only one sixty. [If we take $r_1 = 1.0$, the probability is one in eleven.]

Which scale is the better,—0, 1, 2, . . ., 10 or 0, 1, 2, . . .,5? From the preceding paragraphs one *might* infer that the latter is, because of the greater certainty of a grade. But if we were to use a scale containing only one grade, we would certainly give the true grade every time; and yet we would hardly call such a scale still better. I believe a better criterion is arrived at as follows.

Suppose in assigning grades an effort is made to estimate merit on a scale containing a hundred divisions. Suppose, for example, that two such estimates are 7.5 and 8.4. Then on the former scale they would both be recorded as 8, and they are as widely separated as any two which will be assigned the same grade. What is the probability that the true grades, according to the instructor's standard, for the first is as high as for the second? We have $r_1 = 0.63$, $mr_1 = .9$, $m = 1.4$, and hence P_2 is ¼; the probability is one in four. Suppose that estimates of

grades for two answers are 8.4 and 8.5. Then on the same scale they are recorded as 8 and 9 respectively. What is the probability that the true grade of the first is as high as of the second? It is almost one in two. The chances are nearly even that the wrong answer received the higher grade.

Consider similar cases with the scale 0, 2, 4, 6, 8, 10. Suppose two estimates are 7.0 and 8.9, both of which are recorded as 8; they are as far apart as any two which will be assigned the same grade. What is the probability that the true grade for the first is as high as the true grade for the second? We have $r_1 = 0.63$, $mr_1 = 1.9$, $m = 3$, and hence $P_2 = 1/13$; the probability is only one in thirteen. If the estimates of two are 8.9 and 9.0, the situation is the same as before,—the chances are nearly even that the wrong answer received the higher grade.

It thus appears that on using a scale of only six divisions, 0, 2, 4, 6, 8, 10, we must discriminate very sharply at times, between two answers, but at other times we give equal grades when there is only a small chance (1 in 13) of the one judged poorer meriting as high a grade as the other. In using the scale of eleven divisions, 0, 1, 2, . . . 10, the latter situation is better, the chance is one in four at least that the one judged poorer merits as high a grade as the other.

It appears that the scale 0, 1, 2, . . ., 10 gives a sufficient range for grades on single answers, but that the shorter scale 0, 1, 2, . . ., 5 makes it necessary to record answers as of equal merit when there is quite certainly a distinguishable difference in merit. Accordingly I prefer the former scale.

Grades on Examination Papers. We assume that a typical examination paper contains about ten answers, and that the grade assigned is found by taking the arithmetic mean of grades on the individual answers. The probable error of a grade on a good paper is about 2.0, on the scale 0, 1, 2, . . ., 100.

Suppose that all grades are used. Then what is the probability that when a grade of 79 is assigned, the true grade is as high as 80, the next higher grade? We have $r_1 = 2.0$, $mr_1 = 1.0$, $m = 0.5$; hence $P_1 = 2/5$ (about); the chances are nearly even. We hardly need so many divisions on our scale.

Suppose that only even numbers of the scale are used, that is, that the scale is 0, 2, 4, . . ., 100. Then the probability that

if a grade of 78 is assigned the true grade is as high as 80 is found to be one in four. If the gradė of the paper is estimated at 79.0 and another at 80.9, they would both be recorded as 80. The chance that the former merits as high a grade as the latter is one in three.

If we use the scale 0, 5, 10, . . ., 100, then grades estimated at 77.5 and 82.4 would be recorded as 80. The chance that the former merits as high a grade as the latter is only one in nine. This scale has too few divisions, if we wish to discriminate sharply. If we use the scale 0, 4, 8, . . ., 100 the corresponding chance is one in six; this scale permits of fairly sharp discrimination.

For mathematics papers, I would prefer the scale 0, 2, 4, . . ., 100, as it enables me to make discrimination where discrimination is easily possible. For English papers the scale 0, 4, 8, . , ,, 100 would apparently suffice.

Semester Grades. The most important of all grades is the semester grade, as honors, prizes, scholarships, etc., are awarded on the basis of these grades. It may be all-important to the student.

What is the probable error of a semester grade? It depends largely on how well the instructor knows his students, — how many tests have been given, etc. It seems fair to assume that such a grade is based on the equivalent of forty answers as a typical case. In the more elementary courses this is too small a number in general, while in more advanced courses or in courses where there are very many students in a class it is probably too large. On the basis of this number and the value of the probable error on a grade on a test paper, we have 1.0 for the probable error of a semester grade.

If a grade of 80 is assigned by an instructor for a semester grade, what is the probability that the true grade according to his standard is as high as 85? We find, since $m = 5$, that $P_1 = 1/2500$; it is only one in twenty-five hundred. The probability that it should be as high as 90 is virtually nil, only one in 2×10^{47}. The probability that it should be as low as 79 is one in four.

If grades of 79 and 80 are assigned to two students the probability that the one given the lower grade should have the higher

is one in three, — yet one is virtually recorded in many institutions as 75 (C), the other as 85 (B). If grades of 80 and 89 are assigned, the probability that the one given the lower grade should have as high a grade as the other is not one in a hundred thousand, — and yet they are commonly recorded as equal with a grade of B. We discriminate in one case when the chance is one in three, but do not in another when the chance is only one in a hundred thousand.

There is really a range of ten points for the grade B at many institutions. *The chance that for two students having grades at the extremes of this range the one judged the poorer merits a grade as high as the other is only one in a million.*

If the scale of grades is 0, 2, 4, . . ., 100 then the grades between 79 and 81, for example, are recorded as 80. The probability that a student who is assigned a grade of 79 should have as high a grade as one who is assigned a grade of 81 is one in six. There can be little complaint if we fail to distinguish between such students in grading, but it seems to me that we should not record equal grades in cases where the probability is much smaller that the one judged the poorer merits as high a grade as the other.

One might ask this question: If two students are assigned grades of 80 and 89, what is the chance that if the instructor assigned grades for the same performances on another occasion, independently of the first, the new grade for the first student would be as high as for the second? This is a problem of the third type; we find $P_3 = 1/800$; the probability is only one in eight hundred.

A General Question. How small divisions on a scale of measurements are practically usable? In answer, Starch says,[1] "As a question of psychological methodology the units of any scale of measurements, if a single measurement with the scale is to have objective validity, should be of such a size that three-fourths of all the measurements of the same quantity shall fall within the limits of one division of the scale." If r_1 is the probable error and mr_1 is the unit of measurement, he requires that

$$\Psi\ (mr_1/2) = 3/4.$$

The substitution (7) gives

$$\Psi\ (.476m/2) = 3/4;$$

[1] Science, N. S., 1913, vol. 38, p. 632.

whence, by referring to tables, we find

$$\frac{4769m}{2} = .8134, \quad m = 3.41.$$

He makes the unit 3.41 times as great as the probable error.

Suppose such a scale is used. Then it will sometimes happen that one measurement will be estimated at the lower end of a unit, another at the upper end of the same unit, both being recorded the same although the estimated difference is 3.41 times the probable error of a measurement. What is the probability that the one estimated as the smaller has a true value as great as the other? It is found to be about one in seventeen. The use of a unit of this size thus requires that we shall not at times distinguish between two magnitudes where there is very little chance of a discrimination being unjust.

If we agree that the unit may best be of a magnitude such that in a case like the preceding the probability instead of being one in seventeen should be *one in three, the unit of measurement would equal the probable error; if it should be one in six, the unit would be twice the probable error.*

The best method of recording measurements in general is no doubt to use a unit smaller than the probable error and to record the probable error also. This is not practicable with grades. Personally, *I would prefer to use as unit a measurement at least as small as twice the probable error.* This permits discrimination where discrimination is fairly reliable, but does not necessitate recording measurements as equal when there is little doubt of the existence of a difference in merit.

Principal Conclusions on Grades. To allow opportunity for significant discrimination,

a. Grades on individual answers should be on a scale of about ten divisions,— for example, 0, 1, 2, ..., 10.

b. Grades on test papers should be on a scale of about 30 divisions,—for example, 0, 4, 8, ..., 100, of which eleven are "passing" grades.

c. Grades for a semester should be on a scale of about 50 divisions,— for example, 0, 2, 4, 6, ..., 100, of which twenty-one are "passing" grades.

THE CULTURAL VALUE OF MATHEMATICS

By HELEN E. HOWARTH
Cedar Crest College, Allentown, Pa.

When one receives good news or feels the pleasant thrill after having accomplished a difficult task, his first impulse is to rush to some one and tell him of his new found pleasure. And so it is with me. I have had many happy hours and have experienced a peculiar pleasure which I know I could not have experienced had I not applied myself in the study of mathematics. And because I want my young friends to see the opportunity of experiencing this rare and peculiar and yet wholesome and justifiable pleasure, I am going to try and give some of my impressions as to why a young woman who enters college should pursue the study of mathematics.

First let us consider what kind of a young woman the girl in her teens wishes to become during her four years at college. When she steps off the campus on her commencement day, does she want to be a highly specialized machine, ready to be placed in a certain environment and wound up and set into motion by those in authority, and then work in automatic precision as a result of having learned perfectly the a, b, c's of her specialty? Or does she wish to be a thinking woman, a human being of sound body with a mind so developed that she can meet the problems in life with unrelenting cheerfulness and poise, serenely, with patience and self-possession, and with a spirit which gives her belief and faith in her fellow-beings and in her God and urges her on to work and work with all her strength for the cause of righteousness? She wishes to be the latter and we will call her a cultured woman.

Now let us consider what mathematics is and if it contributes anything toward this development which the young woman desires. Plato called the science of mathematics "divine," Goethe called it "an organ of the inner high sense", Novalis called it "the life of the gods" and Sylvester called it "the music of reason." Professor Keyser says "as an enterprise mathematics is characterized by its aim, and its aim is to think rigorously whatever is rigorously thinkable or whatever may become rigor-

ously thinkable in course of the upward striving and refining evolution of ideas." Mr. Bertrand Russell says "Pure logic and pure mathematics (which is the same thing) aims at being true, in Leibnizian phraseology, in all possible worlds and not merely in this higgledy-piggledy-job-lot of a world in which chance has imprisoned us."

I like to think of mathematics as a special language into which we translate our knowledge. And it is a beautiful language in that it expresses definitely and without a chance of misunderstanding the facts that one wishes to impart to another. It also necessitates a real and true knowledge of the subject under consideration before the facts can be translated. There is no place for the evasive and 'covered' language that has crept into the business and social world and causes much misunderstanding and encourages deceit and flaccidness of character. When one has little knowledge he may use fine phrases to give his hearer the impression that he knows much about the subject under discussion, but he imparts no knowledge. When he really acquires knowledge of the subject and has thought on it so thoroughly that he can translate his facts into mathematical language, he then has power to impart knowledge to his hearer. Scientists, psychologists, sociologists, and engineers continue to work on their respective problems until they can express their discoveries in mathematical language and then they are satisfied that the thinking world understands and believes in their conclusions.

The student in geometry is trained to think deductively and finds that he is able to collect in proper order bodies of knowledge in other fields. He has the power to "frame hypotheses, size-up situations, and deduce conclusions." This is an ability every woman should possess. It makes her life happier; she is more sympathetic and will not fall into jealousy and false suspicions for she has the power to see clearly the causes and effects of events happening around her. Geometry has trained her in the absolute separation of truth and error, and in consistency, and she can mentally tear herself apart from the environment of misunderstanding, confusion, or sorrow, and reason deductively until she comes into full appreciation of the truth and this gives her sympathy and with womanly grace she can justly in-

fluence those around her. In the most commonplace environment a woman of keen discernment is needed for the need of one who thinks is greatest where the need is great.

Since the woman who acquires mathematical ability has power to think rigorously, to size-up situations, and deduce conclusions, to separate truth and error, to understand cause and effect, to sympathize and has poise and self-control and an appreciation of eternal truth, she has developed qualities which give her belief and faith in her fellow-beings and in her God. And she has developed patience, for, as Professor Keyser says "from time immemorial, there has been but one way to become a mathematician and there will never be another: it is a way interior to the subject and involves years of assiduous toil. Short cuts to mathematical scholarship there is none, whether the seeker be a philosopher or a king."

My appeal to the young woman entering college to study mathematics is not only for the character development and intellectual attainment (which of course will also benefit the many whom she will influence) but it has another phase. As a vocation, the teaching of mathematics is highly desirable. Our schools need teachers of mathematics who have realized that spirit in which it should be presented, the cultural aim, and the high place it has in the development of the student's intellect and character. The manipulation of symbols may be taught by many teachers, but mathematics in its full significance can be taught only by such a woman as I have tried to describe. In reference to mathematical symbols let me quote:

> "They are things of so frightful mien
> That to be hated need only to be seen
> But often seen, familiar with their face,
> We endure them first and then embrace."

Teachers are wanted who can lead the students to the very last word of the fourth line of the above quotation.

There are really few normal minds which cannot acquire a fair mathematical ability. This has been proven in many cases. Of course a lazy intellect or one working in opposition to the will cannot progress mathematically. A normal mind with a willingness to work and a desire to think to its fullest capacity may

acquire the treasures and peculiar pleasure in store in the science of mathematics. When I hear a person of splendid training and ability in another field of thought, speak lightly and shamelessly of never having had any mathematical ability, a sense of pity for her misconception overwhelms me. And when that person says "what use is mathematics anyway?" I become speechless with sympathy for her at the lack of understanding she displays. A great mind endeavors to investigate and understand, as far as time allows, all possible fields of thought. If time or inability does not permit a mind to master a particular field, should the person try to excuse herself by belittling the personally unexplored field? This is a weakness in character. What we need is more appreciation of the worth of others. This idea is told in a simple way in the following story written by a high school pupil:

> It was a clear brisk day in March and Mr. Trigonometry was sitting in his study. A large open book lay on the table before him, but he was not reading—no—he was thinking.
>
> "Yes, my history has been wonderful. I have derived a famous inheritance from the Arabs, Hindoos, Europeans and even the ancient Greeks. All through the centuries my house has prospered and now I find myself recognized with importance in the New World. When—Hello, brother Geometry, I am glad to see you this afternoon. How are you?"
>
> "I am happy, brother, and am glad to find you at leisure for a few moments. What is it I hear about a meeting this afternoon? I hope no one comes very soon."
>
> "There is just time enough for a splendid chat before we shall have the pleasure of entertaining a few important factors in the three dimensional space."
>
> "Whom shall I have the honor of seeing?"
>
> "Messrs. Latin, History and English. Have you ever met them?"
>
> "Yes, we have been thrown together in various circumstances and have often found it confusing because we do not know each other very well."

"It is fortunate that you are here, for they are coming to my house this afternoon so that we may become better acquainted. It happened last Monday, while I was at an assembly conducted by Mr. Physics, that each of the gentlemen were called upon to speak. It was extremely embarrassing because each had been accustomed to conversing with his own relations and hardly knew how to make the rest of us understand him. Although Mr. English succeeded best of all the speakers, it was he that suggested that we meet more familiarly. We had a discussion about a meeting place, and, of course, each wanted the meeting at his house. After considering, we found the disposition of Mr. Latin, Mr. History, and Mr. English were rather changeable, so we agreed that the meeting would be most positive and accurate at my house. All the plans were made after Mr. Physics had gone, for we would rather not have him in our midst because he is so very precise. And then, too, when we do not believe in something he believes; he says, 'I will show you by experiment.' This always offends Mr. History because, as he can never prove anything he says, he thinks things should be taken on faith."

"I think it is a splendid plan to have such a meeting," said Mr. Geometry. "I hope I shall be understood. I am not usually welcomed in company. Most people think me queer and do not wish my acquaintance. However, when anyone discovers my real disposition, we become the dearest of friends and remain inseparable forever. Is that Mr. Latin I hear coming now?"

As he spoke there appeared a dark, heavy man, slowly advancing. Mr. Latin had hardly been seated when Mr. History paced steadily into the room. Then lightly, yet sedately, Mr. English hurried in.

"It is a delightful day for our meeting, friends," began Mr. Trigonometry.

"What pleases me," said Mr. English, "Is that we are all here in this beautiful new world, America. My ancestors had to struggle for the lead in this country, but now I am universally important."

"What you say is true," assented Mr. History. "Everyone in the land honors you. However, if it were not for my ancestors none of us would be in this land."

Mr. Trigonometry coughed.

"We owe a great deal to my father Columbus. If he—

Mr. Trigonometry coughed again.

"Yes, it was through him that the Atlantic was crossed."

"Yes, he first crossed the ocean," said Mr. Trigonometry, "and he had with him the ephemerides calculated by my father Regiomontanus. It is to the great Regiomontanus, brother Geometry, that we are indebted for our former union. Although we have since his time been separated, the union was the beginning of our respective importance."

"I have heard from my parents," replied Mr. Geometry, "that my ancestors welcomed your ancestors most cordially into the family. And then in later years, after you had become strong, it was considered by both parties to be for the best that each should go into the world independently. I believe, Mr. Latin, that your ancestors were pragmatic. Do you know whether they were acquainted with mine?"

"My ancestry is noble," said Mr. Latin. "I do not remember hearing much of their acquaintaance with your family. It has occurred to me, however, that they rather disliked one of your ancestors, the root-extractor. My fathers have been famous for their bravery and courage, yet they found Mr. Root-Extractor difficult to master. There are many names, such as Cicero and Virgil belonging to my family which have made Mr. History renowned because of his acquaintance with them. The Latins were also visited by many other families. Even Mr. English, who now predominates in this land, has some ancestors formerly of my family."

"Each of you has played an important part in bringing us to this wonderful land," said Mr. English. "And, although I cannot trace so much importance in my ancestors, I know that none of you would be very popular in this "New World" if it were not for my utmost importance to everyone in the country. I have heard people speak of you, Mr. Latin, as dead. Understand, however, that the

Latin spirit still hovers over the country and in truth many have said that after understanding you, I am really more readily understood."

"I am sorry to leave my worthy friends," said Mr. History, "but I have an appointment in Panama which must be seen and recorded."

"As the sun is setting," said Mr. English, "I had better go out and meditate and find some new expressions of description."

"I, too, must be going," said Mr. Latin, "for I must prepare to visit my young friends who burn the midnight oil."

So saying the three departed. Ever after, each of the honorable Messrs. Trigonometry, Geometry, Latin, History and English realized the value of the others and appreciated his own worth and importance in the powerful country of America.

. . . .

I have not told you of the practical justification of studying mathematics such as the dependence of science and inventions of our great industrial and commercial enterprises and of the beauty in great architectural constructions upon mathematicians. It is not my purpose here to tell of these things. I simply want to give suggestions as to how mathematical understanding contributes toward the making of a cultured woman and enables her to rise above narrow and shallow ideas concerning her environment into a beautiful belief and faith in God and man, and gives her pleasure through the power to think and appreciate truth and beauty of ideas.

DEFECTS REMAINING IN THE NOTATION AND NOMENCLATURE OF ELEMENTARY MATHEMATICS

By JOSEPH V. COLLINS
Stevens Point, Wisconsin.

A source of difficulty and confusion in learning algebra and geometry is found in their terms and symbols. Most terms are Latin words and some symbols are hieroglyphic like. It will be seen that a considerable number of the suggestions of the Committee on Terms and Symbols amount to replacing foreign phraseology by English. The work of the Committee whose report has been recently printed is worthy of all praise. Practically every proposal will commend itself to the thoughtful teacher.

Naturally such a Committee would want to proceed along conservative lines so as to avoid controversy and secure a maximum of practical results. Doubtless numerous proposals were made concerning which no decision was reached. It is proposed here to consider some important points, not touched by the Committee, in the hope that their consideration will prove of real help to teachers.

A study that has interested mathematic teachers has been that of common errors and the frequency with which they occur. This work is well enough in its way, but should lead to the elimination of these errors rather than their mere avoidance. Formerly physicians doctored symptoms; now they seek for specific remedies.

The reader will be more interested if he knows in advance that such questions as the following are being considered. Why is it that pupils so often drop denominators in addition of fractions, and retain them in the solution of equations? Why is it that pupils can not tell, when every factor in a problem in cancellation cancels out, whether the answer is 0 or 1? Why is it that pupils frequently do not know the value of $(\sqrt{x})^2$, and having the matter explained to them fully, promptly forget the answer the next time the question appears? Why is it that radicals is such a difficult topic in algebra? Why is it that pupils have trouble in understanding percentage in arithmetic and proportion in geometry?

The notation of elementary algebra and geometry represents only a small part of that of mathematics, as anyone can see by consulting such a work as Peano's. Still it is fundamental and on it is based all that follows in the manifold developments of the great field of mathematics as known to the modern world. For this reason, as well as for the educational one, there should be no flaws in either its notation or nomenclature.

The following principles for testing a notation and nomenclature are self-evident:

(1) *Two essentially different fundamental concepts, operations, or relations should not be represented by the same symbol or name.*

(2) *The same fundamental concept, operation, or relation should not, in general, be represented by different symbols or names.*

There are exceptions to the second principle, as in the need for the various grouping symbols, parentheses, brace, etc. It may be noted here that addition has but one sign, subtraction one, multiplication three (counting juxtaposition), and division four.

Perhaps the most glaring instance of the breaking of the first principle is found in the use of $=$ to denote both conditional and identical equality.

The symbol $\equiv$ has been gradually creeping into mathematical books, chiefly higher mathematics, but the step has never yet been taken of uniformly employing it for all identical equations. Were this done the symbol would indicate that a quantity, (not a relation between two quantities) is given, whose form is changed but not its value; that denominators must be retained, and that letters in such identities can have any values. The educational importance of the distinction between identical and conditional equations was first called to the writer's attention by the report of the Committee of Fifteen of the N. E. A.

An unfortunate instance occurs of the use of the same name and symbol for the two concepts of indefinite straight line and straight line segment. Arguments for the employment of the word sect for straight line segments are cogent, but the Committee seems to have followed the wider usage and recommended segment.

It remains to call attention to two entirely distinct operations which are almost invariably denoted by the same mark and called by the same name.

These two operations are cancelling equal factors in dividend and divisor and the so-called cancelling of equal quantities with opposite signs. . Mr. L. P. Jocelyn's Committee in its Report to the Central Association on a Uniform Notation in Mathematics and Science favored the use of the phrase "destroy each other" in place of "cancel" for the latter operation, but did not suggest the use of another symbol to denote it. . It is here recommended that an $\times$, cross be drawn through each of two quantities that destroy each other, following the custom often employed in long division.

Let us turn now to cases coming under the second principle in which two or more symbols or terms are used for the same concept. We have, as an example, the terms nought (naught), cipher, and O(oh). Of these there is a prospect the last will supplant the others, as seems to be the case now in the business world. The use of O for both zero and nothing seems to lead to no confusion. Nought is ill-sounding, long and hard to spell, and is pronounced incorrectly "ought" by a large part of the population. Its disappearance from mathematics may well be desired.

The dot to denote multiplication has its defenders and opponents. Both Committees declare against its wide use in this country, because of the danger of its being mistaken for a decimal point. The other symbol for multiplication is objectionable because it is given two distinct meanings, "times" and "multiplied by." How to dispel this fog has so far proved an unsolved riddle. Perhaps the use of a large $\times$ for "multiplied by" and a small one for "times" might prove a solution of the puzzle. Perhaps also the small $\times$ could take the place of the dot. If anything can be done to save the poor urchins from getting into so many scrapes with their teachers on account of a confused use of "times" and multiplied by", it certainly should be done.

The most unfortunate of all duplications of symbols for the same operation is the use of both the radical sign and the fractional exponent for roots. The double notation makes the study of radicals thrice as difficult as it otherwise would be. The frac-

tional exponent is the natural notation. Unfortunately the radical sign had been introduced a century before Newton thought of fractional and negative exponents. The radical sign notation is easier to write in many cases than the fractional exponents. This is because the numerator 1 and the denominator 2 of the fractional exponent are never written in the radical notation. If this advance were made in the use of fractional exponents, (leaving only the fraction line, horizontal or oblique) one notation would be as easy to write as the other. In the calculus and for certain reductions in radicals, the fractional exponent notation is essential. Chrystal in his Algebra limited the meaning of the radical sign to denoting the arithmetical root, but the Committee recommends that both notations denote only the arithmetical root. Presumably this limitation is for elementary mathematics.

Compare the following forms, the first used continually in the solution of radical equations:

$(\sqrt{x})^2 = x$ and $(x^{1/2})^2 = x$; $(\sqrt[3]{(x+2)^2})^2 = \sqrt[3]{(x+2)^4}$ and $((x+2)^{2/3})^2 = (x+2)^{4/3}$

The writer believes the time is quite sure to come when only the fractional exponent notation will be used.

Probably the difficulty experienced in learning proportion is due in considerable part to the two different ways of writing a proportion and the three ways of reading it. The Greeks taught that a ratio and a fraction are not the same. We prove that a fraction can always be found differing infinitesimally from the ratio, and then treat proportions as the equality of quotients, but retain the Greek notation and nomenclature. If we followed the *tendency* of the Committee, we would drop the word ratio as meaningless to children, and substitute for it quotient. One can hazard a guess that if all the High School and College graduates in the country were asked to give a precise definition of ratio, a tabulation of the results would be startling.

There is no fault to be found with the word per cent, or the notation for it, except when the word is translated by anything else than *hundredths*. A hint as to why percentage presents difficulties may be had from the following:

$$62\tfrac{1}{2} \text{ per cent} = 62\tfrac{1}{2}\% = \frac{62\frac{1}{2}}{100} = \frac{625}{1000} = .625 = \tfrac{3}{5} = \text{etc.}$$

When it is considered that a reduction from the first form to any one of the other five and the converse operation, and a reduction from the second to any one of the remaining four and the converse operation may be called for by some problem at any time, it is apparent that the subject is not as simple as it looks, particularly when one remembers that this takes no account of the solution after such a reduction has been made.

The preceding ought to show with tolerable clearness how intimately connected the terms and symbols of mathematics are with the teaching of the subject.

The question arises what can the teacher do to improve conditions in addition to following the suggestions of the Committee? If the teacher feels the need of something being done, and is willing to make the necessary effort, the following are suggested: (1) Require the use of $\equiv$ for identical equations put on the blackboard or in written work, either part of the time or all of the time. (2) Always use the term "destroy each other" of equal and opposite quantities, employ $\times$'s to cross them out. (3) Tell classes that fractional exponents provide the logical notation for roots: how, historically, the radical sign came to be used: and that the numerator 1 and denominator 2 were to be understood when not written (the fraction line only being written), this notation would be preferable to the radical sign. (4) Always write proportions in the form a/b = c/d, and read them at least part of the time the quotient of *a* divided by *b* equals the quotient of c divided by d, explaining the historical reason why "is to" and "as" came to be used.

In conclusion it can be said the world owes a great debt of gratitude to the European mathematicians of the 16th Century for developing our algebraic notation in almost its present form. That there would be blemishes in it was naturally to be expected. To have these blemishes remain permanently is unthinkable. Social and political reforms are a commonplace. Then why should the notation and nomenclature of mathematics remain static at any defective point.

THE LOGIC OF MATHEMATICAL PROCESSES[1]

By HUGO F. SLOCTEMYER, S. J.
St. Louis University.

It is my privilege today to speak to you on a subject in which I have been long and deeply interested. Unfortunately for me, but possibly fortunately for the profession, I am not a teacher of Mathematics, though at times in my career through force of circumstances, I have engaged in that important and absorbing occupation. I am and have been occupied for some years in the teaching of Physics in colleges and engineering schools, and therefore, although I have not had the opportunity of making observations on the teachings of Mathematics from the standpoint of the professor, I have had occasions innumerable of calculating the results of their teaching in students.

These results are as varied as the flowers in Spring. With a quickness of perception and a passion for accuracy that is most commendable in a scientific man, some students give unmistakable evidence that their elementary training in Mathematics has given them that capacity for coherent, logical and precise thinking, which it is the enviable province of Mathematics to impart. They trace a line of reasoning with unerring sureness; theories and hypotheses remain such until by the overwhelming evidence of theoretical or experimental research they assume the character of reality; quantitative relations are dealt with and computed with a refined sense of correctness and detail. These are the students to whom Mathematics has been a trainer of the intellect. These are they who have reaped the benefit of much drudgery and painstaking toil, both in the discipline of their mental faculties and in their practical application of their knowledge to engineering problems.

But, and this is the other side of the picture, there is a large number of students who give equally unmistakable evidence that their training in Mathematics has been as inefficacious as it has been without purpose. These men can not employ their Mathematics for the intention for which it was incorporated in a course of studies. They are helpless before a problem of unusual procedure. This is usually the most telling evidence of weak training. But deeper than this and more comprehensive in its

[1]A paper read before the St. Louis Mathematics Club, February 11, 1922.

effects, is their inability to pursue a course of clear, concise reasoning, from its inception in a few scattered facts to a conclusion, widespread in its application and solidly founded upon premises knit together with the thread of unbroken sequence. Here is their real want. Here is the chasm which nothing can span. This is the void which will be a detriment to them in after life, an absence of logical, mental efficiency that can be replaced by no other asset. This is the toll demanded by incapacity in teachers or by inattentiveness and carelessness on the part of the student, or by both.

That such inefficiency and uselessness is widespread is certain and is attested by experience. It seems that some of our teachers lose sight of the real purpose of mathematical training, which is suggested by the nature of the subject and which we should strive to realize, namely the imparting of a capacity to think logically and clearly and coherently, with accuracy and precision. It is no mean accomplishment and the lack of it is traceable not only in the world of mathematical thought, but in the sphere of all mental activity that mankind must of necessity perform. Our minds are logical by nature. Man demands a connection between parts of a system of thought. Disconnected facts mean nothing to an individual who is searching for the truth and since the mind of man loves the truth and desires it, it must necessarily prize and value the art of clear thinking, by which truth is obtainable. It is within the province of Mathematics to give that facility and the part of Mathematics in education to train the mind of the student to use its facilities to the best advantage.

If Mathematics is to give a capacity for clear thinking according to the laws of right reason, it itself must be logical. It is this particular fact that I wish to emphasize with you on this occasion. Mathematics is a branch of knowledge in which the laws of rigorous thinking are most universally in evidence and hence it is the enviable duty of the teacher of Mathematics to insist upon that fact and to make it the vehicle of a training for the student which no other subject will impart.

What is meant by saying that Mathematics is logical? What is logic? Logic is the art of correct reasoning. It lays down right rules, which, if the mind observes them, will lead the mathe-

matician to correct conclusions. It defines criteria of evidence for the establishment of a truth. Whether the conclusions of the mind are correct or false will depend upon how they respond to an examination set them by the art of logic. It deals exclusively with the mind and with its operations on what is presented to it. It does not concern itself with facts, it cares not for data or observations. It takes all these for granted, as established or not established. It does care for what use is made of such material. It is concerned with what the mind concludes is right from all the mass of evidence that is presented to it. Therefore since mathematical knowledge is a matter of induction or deduction, the whole of scientific knowledge is fit for examination by this all-embracing and scrutinizing standard. Logic therefore is an art which has to do with the entire range of human activity, be that directed to the increase and dissemination of knowledge or to other concerns of life. It is the one necessary adjunct to intelligent action which is required and is sufficient to establish man as the king of the created universe. Mathematics then is pre-eminently logical.

There are two chief methods of reasoning. One is called the Inductive Method, the other the Deductive Method. In the former we proceed from the known to the unknown by inference. We notice that a certain fact is true of a phenomenon. By repeated observation that same fact is seen to be true of many other phenomena of the same sort. By inference we come to the conclusion that the same fact is true of all such phenomena. It is an inductive process of reasoning, a process of inference, that brings me to that conclusion. The premises are facts, the conclusion is a truth. Induction is a process of generalization and who will say that Mathematics is not a process of generalization? We argue from a few members of a class to a whole class.

The methods of deduction is more complicated. It establishes two premises either explicitly or implicitly. A conclusion is then drawn. The premises are such that if both are admitted, the conclusion is inevitable, as, for example: Man is possessed of a reasoning faculty. I am a man. Therefore I am possessed of a reasoning faculty. The first statement may be of the nature of a definition, which must be established. The second statement may be the result of experience. It makes no difference by what

way we establish the first two statements. IF THEY ARE ESTABLISHED the third statement is true and can never be false as long as the first and second remain verified. That is the method of deductive reasoning and it is precisely in virtue of this and the inductive process that Mathematics is so rich in its heritage of training power.

Mathematical truth may be said to be of two kinds, not entirely distinct. The character of mathematical reasoning that I follow regarding the deflection of a beam of light through a prism is far different from that regarding the division of a rectangle into four equal triangles, by having recourse of the proposition in Geometry which states that the diagonals of a parallellogram bisect one another. The first process of reasoning depends upon whether a prism exists or not. The measurement which I use as data are the result of experience and therefore if no prism exists, all that I predicate about the deflection of light through it and the mathematical reasoning dependent thereon, is vain and useless language. Whereas the truth that the rectangle can be divided into four equal triangles does in no way depend upon the existance of a wood, cardboard or metal rectangle. Moreover, that the diagonals of a parallellogram bisect one another is true whether there are any parallellograms or not. These are metaphysical truths and so, Mathematics, which we designate as Pure, in contradistinction to Applied, is of the nature of metaphysics and is the highest and purest kind of truth. Two times two are four and three times three are nine, whether there are two or three units of the same material in the world to multiply together or not. That truth has nothing to do with the material world about us. We use it in daily contact with the material world, but that is because it has a universal application. The fact is true, because since multiplication is a shortened form of addition, two added to itself twice will yield four and three added to itself three times will give us nine.

The example chosen above is simple enough. In other of our mathematical laws, to which we direct the attention of our students, the truth is not so evident. Let us consider for a moment the law of signs in algebraic multiplication or division. We are told that to multiply two quantities of like sign will yield a positive result, whereas if two quantities of unlike sign are multi-

plied, a negative product will follow. The same, with proper changes, is predicated of division. Again, we add quantities of like signs by adding the absolute values of the numbers and prefixing the common sign: if the signs are unlike, we are told to take the difference between the absolute values and prefix the sign of the larger. These are very simple working rules, but surely unintelligible from the standpoint of the reasons underlying them. Are they true? We are told that they are. Can we go through a process of reasoning that will result in the fact, for example, that we add correctly quantities of unlike signs by finding the difference of their absolute values and prefixing the sign of the greater? It is an inductive proof, beginning with the very idea of unit quantity and the arithmetic series, and by slow accretions of facts, we can arrive at the above-mentioned statement of a rule. Students work by the rule and know nothing of the theory that underlies it. To substract we are told to change the sign of the subtrahend and then to add. Is that true? It gives results that can stand the test of scrutiny and so we accept it. But underlying that rule which the student in the class room blindly works with, is an inductvie process of reasoning that makes the statement a safe road to the correct answer. Now, it is this precise substratum of logical reasoning that makes of mathematical work the great mind trainer that it really is.

Let us take a simple example. Suppose that we wish to add (plus 2) and (minus 3.) The rule states that we are to take the difference of the absolute values and prefix the sign of the greater. The result is (minus 1) as the sum of (plus 2) and (minus 3). What is the reasoning process that establishes the rule as a safe guide to correct results? Let us begin with this statement: (plus 2) and (plus 3) equal (plus 5). This can be proven by having recourse to the nature of and a very simple operation with the arithmetic series. If we add to (plus 2) a number less by one than (plus 3), we shall get as a result a number less by one than the sum of (plus 2) and (plus 3), namely, (plus 4). Hence 4 represents the sum of the units in the symbols 2 and 2, and is the symbol next on the left from 5 in the arithmetic series. If (plus 1) is added to (plus 2), the number to the left of (plus 4) is gotten, namely, (plus 3). If 0 is added to (plus 2), 2 is gotten as the sum. Add one less than 0, namely,

(minus 1) to (plus 2) and the number to the left of (plus 2) is the result, namely, (plus 1). Thus the process can be continued: (plus 2) plus (minus 2) equals 0; (plus 2) plus (minus 3) are (minus 1). This is the result looked for, (minus 1), as the sum of (plus 2) and (minus 3). From this consideration a prescriptive rule can be established by direct inspection to produce the same results as this reasoning process and much more quickly. That rule is: when one number is positive and the other number is negative, the SUM of the numbers will be their *difference* in absolute values and the sign of the sum will be the sign of the greater in absolute value. Hence the rule is nothing but a "short-cut" to the same results as are achieved by an inductive reasoning process, which in turn is established after a sufficient number of individual operations have been studied.

What I have said about the elementary processes of Algebra, can be said of every mathematical process. The truth is there: it underlies methods and formulae and it is the neglect of this feature of mathematical training, which my very limited experience forces me to believe, is the leakage point through which much of the value of mathematical work is lost. Practical men work by formula. Engineers will tell us that they care not for the way the rule was developed nor the manner of growth of the formula. If the rule gives results and the formula "works", they are satisfactory to the mind of the practical mathematician or scientist. But for the student, he who is to secure from a course in Mathematics not merely the utility value but also the cultural value of the subject, it is not enough to know that the rule is correct or that the formula will give satisfactory results. He will profit much by knowing how it was developed and by giving close attention to the process by which it was evolved. For him the rule is but a sure way to the same result as is achieved by following the inductive or deductive process by which the effect was established. The fact is anterior to the rule and is true independently of the rule. We add two numbers algebraically by subtracting their absolute values and prefixing the sign of the greater, *not* because the rule prescribes that method but because there is an inductive process by which the same correct result can be gotten, of which process the rule is an equivalent statement. The authority of the rule lies in the logical process

that gives the same results. It is in these processes that the mental acumen of the mathematical world has been exercised. It is in these derivations that the real value of Mathematics as a mind trainer is to be found.

I do not say that it is always possible to teach all grades of pupils the mental intricacies and gymnastics that the founders and developers of Mathematics have performed, to establish our laws and formulas. But it is possible and necessary for the teacher of Mathematics to understand them, to keep them in mind during his teaching hours and to insist upon them with a class as a demonstration and as a mental exercise, of which no account in an examination need necessarily be given.

What is to prevent me as far as the capabilities of the student are concerned, from delving a little deeply in treating of simple equations, into the real reason why it is permissable to transfer a term from one side by multiplication to the other side by division, or from one side by addition to the other side by subtraction? Surely the student gains very little from knowing that it IS permissible and resting in ignorance thereafter. Or, why in proving a proposition in Geometry, may I not spend some time in developing the structure of the proof? I might with profit go into details showing how the proof is made up of parts, the first being logically necessary for the second, and the second being required before the third is possible. So I might show how a proposition already learned is not applicable in this proof, how it is not wide enough in its comprehension, how if it were quoted as a proof for a step in this, the proof would be faulty. In this way I will insist upon the logical character of the study. In this way will I bring home to the mind of the pupil the all-important fact that there is in Mathematics a logical character, a continuity of thought, a sequence of development, that binds the subject into one beautiful whole, beginning with the fundamental concept of unity and extending through the maze-like theories of numbers with which the research mathematician deals. Indeed if that be done, Mathematics will do its allotted work in the curriculum of our modern schools. If it is not done, Mathematics shall fail in its main and most valued purpose.

With the logical character of mathematical processes emphasized in the class room, the mind will be trained to clear and cor-

rect thinking, which will redound to its own benefit in all departments of life. To be able to think coherently, to be able to draw correct conclusions, to lay down premises which are not hypotheses or theories and from them to derive conclusions which will stand the test of logical scrutiny, are indeed powers, which are well said by able educators, to be the real objects of mental training. To have a logical mind is to be able to ferret out the truth and to avoid error. And what greater capacity can an education give a student than these two powers? To seek truth! Would that we were all able to seek the truth in daily life, in principles of conduct, in attitudes of mind, in social relations, in politics, in business life. What a world this would be if every man acted logically? To avoid error! This is another much to be desired object. Error in civil and private life is the one source of the difficulties of human society. Errors in judgment, upon which our conduct is based; errors in points of view, which argue a want of a sense of values; errors in our ideals and ambitions, which mean a lack of perspective. Indeed if this thought be pursued, it will become evident that the mind of man is not functioning according to the will of its Creator unless it function logically. For with clear, correct thinking, the truth is gained and without it we fall into the grossest fallacies. To those of you whose primary duty it is to teach the science of Mathematics, I say: "May God prosper the work!" for you can not possibly be engaged in an occupation more prolific of good, more solidly founded upon the necessities of the student and more promising for his material and spiritual welfare.

THE WAY MATHEMATICIANS WORK.

JAMES BYRNIE SHAW.
University of Illinois.

It is of interest to those who study mathematics and also to those who teach it to note what mathematicians have had to say as to the manner of arriving at their results. Mathematics is a constantly growing part of human knowledge, and has the unique quality that its results are permanent, and form a synthetic unity which is not paralleled by any other part of human knowledge. The text for this paper is found in an inquiry instituted in 1905-6 by the journal "L'Enseignment mathématique" as to the method of mathematicians, and which was discussed by Flournoy, Professor of Psychology, and Claparède, Director of the psychological laboratory, in Geneva. While one must not over-value the results of such a questionnaire, yet they have their place, and may at least suggest some things to reflect over. This summary and discussion was published in 1905 and 1906, and in 1908 as a pamphlet, by Gauthier-Villars, Paris.

We may pass over as not of present interest questions as to the hereditary character of mathematical ability, and its early manifestation. The next questions relate to the branch of Mathematics that invited the interest of the respondent. There are eighty-two answers, of which ten are to the effect that their interests were in the general side of abstract mathematics, such as the logic and method of mathematics, twenty-four were interested in algebra and analysis, twenty-four in geometry, fifteen in geometry as connected with analysis, and nine in applied mathematics. The detailed replies probably show what in larger numbers would be similar responses, the overwhelming majority of mathematicians being more interested in the development of the subject for its own sake than for any applications. The answers repeatedly allude to the fascination of the abstract side of the subject and we will find answers to other questions pointing to the same esthetic appeal. The obvious deduction for the teacher is to be drawn, for if persons become mathematicians on account of this appeal of pure mathematics, it is no doubt true that students also will be attracted to the subject and be interested in it for the same reason. It is evident that applications should not

be ignored, but it is very doubtful if an emphasis of the applicational side will incite students to become mathematicians. In the latter decades, under the stress of vocational training and utilitarian ideas, there has been an increasing amount of "application" inserted in the text-books, often drawn from engineering and other subjects the student knew nothing of, and had little interest in, and the effect in increasing interest in Mathematics, so that the student goes on to study higher mathematics, is not noticeable. There is no end of opportunity for the real mathematical teacher to make Mathematics fascinating on its own account, and then the applications can follow naturally.

The next two questions asked for information as to how the elementary part of mathematics had been studied, and what difference had arisen in pursuing the more advanced parts. The striking fact in the majority of answers is the freedom that was enjoyed, and in many cases is insisted upon as necessary, in becoming acquainted with mathematics. There are many replies to the effect that set courses, or the attempt merely to learn what others had done, was fatiguing, and uninteresting. In pursuing the subject further most of the answers indicate that personal tastes and desires dominated the choice of subject. This does not mean that in the elementary schools no fixed course was followed, for it is obvious that the elementary student could not be left to his own devices. But it does indicate that a certain amount of freedom should be introduced early in the college course, and in the work of the Junior and Senior years the student should have a chance at several different lines of study. And in these he should be encouraged to do independent reading and research even if he did not complete the prescribed work of the course. There are many beauty spots in the land of Mathematics, and it is doubtful if personally conducted tours are ever equal to those in which the individual wanders at his will and finds his beauty himself. An artistically designed program may look fine to the committee that works it out, but it has not the merit of the program the student makes for himself. The controlled program is still more to be condemned in graduate study, but this fact is so evident it need not be emphasized.

The group of four questions that follow next inquire as to the role of the unexpected, of inspiration or intuition, and of the sub-

conscious, in mathematical discovery. The replies indicate that a certain amount of study, reflexion, incubation of the problem is usually necessary, sometimes more prolonged, sometimes less, but without which nothing would be achieved. In short if one never sits down to think about mathematics, he will never get any results in it. Mathematical theorems do not float around like thistle-down, alighting by chance where never wanted. But a good number who seemed to catch the real essence of the question insist upon the unexpectedness, the apparent spontaneity of the new idea. The term inspiration is variously understood, as for instance: ideas that try to enter the world; a mysterious function of the individuality of each person; the mental process, impossible to be observed, by which from a series of phenomena one passes to their law; the ability to leap the gaps and chasms in a given domain; a sort of fluorescence of previous impressions; an instinctive presentiment of truths or new methods; imagination; the work of unconscious cerebration. The question savors of the old one as to whether genius is merely the capacity for prolonged hard work. It is granted that the hard work is needed, but alas if the happy thought never spreads its butterfly wings! Every mathematician has had the experience of pounding out a long hard demonstration of some theorem he has thought of, and later have suddenly appear a far more direct and elegant proof of the same theorem. This latter is what was meant by the question.

The reflection that occurs to us at once is that if a method could be discovered by which all students could be taught how to go to the heart of a problem and see directly its conditions and relations, what wonderful progress they would make! But when the usual instruction is based upon a set of routine procedures, upon carrying out every case that may arise in the general way, of learning "how to do this, how to solve this sort of problem" what may we expect in the way of originality and incisiveness, of freshness and spontaneity? In set courses, in merely learning and being examined upon what others have thought, what chance is there for the development of the intuitive power? The power of concentration must be acquired, and pedagogical methods have sufficed to bring this about, but the mysterious power of conceiving an idea must be also acquired or developed, and pedagogical methods have no conspicuous success along this line. The imag-

ination is the source of this power, and methods have yet to be devised for systematic training of the imagination. Deficiency in imaginative power lies at the base of many a ruin in mathematical study. A good course in poetry might produce better mathematicians than one in physics, or logic. The methods of teaching that undertake to develop the reasoning power, do, and should be expected to, fail to produce mathematical ability. Theorems are not produced as the result of a series of logical deductions, but the theorem is usually surmised first and proved afterwards. This is what is meant by inspiration, intuition. So far there seems to be no adequate way even to conserve what native power students have in this direction.

There are two questions that relate to whether in investigations the procedure is sketched in notes, or whether it is at once developed in full, and if there is any difference in the method of invention and that of presentation. As might be expected the flow of ideas, when it comes, is too rapid to be written out in full, and notes only are kept in most cases. The mode of presentation is then considered with meticulous care. The suggestion is that it is worth while to have students produce outlines of methods or proof or of attack on their problems, as frequently as definitely worked out, detailed papers. The chance to see through a solution without the labor of having to give all the details, would be of great value to the beginner, for it is what he really must learn to do if he purposes to become a mathematician. Indeed this fact has been recognized by some successful teachers.

A group of three questions relate to the importance of becoming familiar more or less with the literature of a problem before beginning to work on it. The answers are quite uniform to the effect that the beginner should become somewhat acquainted with the literature but should not make the mistake of trying to read it exhaustively. Above all he should not delay long his own attack, even if what he discovers turns out to be already known. To this end he should select a few of the most important contributions to his subject, become familiar with them, then undertake to handle his problem alone. His general reading should really be for a different purpose, namely to fructify his mind, so that suggestions will come for investigations of his own.

A group of questions relate to the amount of time that is spent in investigations, whether it is continuous or at intervals, daily or otherwise. As might be expected great variety is found here. Some become possessed with an idea and labor continuously till it is accomplished, then take a vacation for a longer or shorter period till another idea arrives. Few find that a regular program of daily study accomplishes much. One reply says that to be a true mathematician one must give himself up to it heart and soul, and all the time. The important feature inthe answers is the absorbing character of the subject itself, so that from the monotony and drudgery of daily occupations, one may turn to this for real relief, relaxation, and play. It is the character of fascination again.

Finally we consider a group of questions relating to the diversions of mathematicians. These are to a considerable extent of an artistic nature. Many who do little themselves are nevertheless interested in music, poetry, painting, art in general. Going with these are such diversions as walking, bicycling, golf, etc. Many find pleasure and profit in these physical pursuits. Some are interested in intellectual discussions, in natural science, and find in such diversions, recreation, rest, and recuperation. The conclusions we draw are that first of all the mathematician should cultivate as far as possible any artistic taste he may have, and secondly that he does well to have a wide range of interests in life in general, and be able to find himself at home with almost any educated companion. There should be no narrowness in specialisation, for one may become an intense specialist and yet maintain his perspective, his sense of proportion as to life and knowledge in general. This the mathematician should strive to do. The reaction upon his own subject will be beneficial and he will be better able to keep his balance in it.

There are some other questions as to the mode of life of mathematicians but they do not relate directly to the topic considered. This questionnaire might well be duplicated nowadays in a wider manner, although one of the psychologists suggests that long questionnaires seldom get valuable attention and responses. A sort of "test" might however be invented which would show some of the salient characteristics of mathematicians, with a direct reaction upon the pedagogy of the subject.

INFORMAL TESTS FOR DIAGNOSIS AND REMEDIAL TEACHING IN MATHEMATICS

By PETER L. SPENCER
University High School
University of Oregon, Eugene, Oregon

If we should ask the average person which subject now being taught in our schools lends itself most readily to testing, he would probably say without much hesitation, "Mathematics". There seems to be a common opinion that one can easily devise, administer and score a test in mathematics and can tell thereby very accurately just how much the person examined knows of that subject. Just why people have such a notion is hard to determine but any teacher of mathematics knows that it is far from the truth. It takes just as much thought and just as much experimenting to devise a useful test in mathematics as it does for any other subject, in fact a very plausible case can be made to show that it takes considerably more.

The writer, being a teacher of mathematics, is interested in the problem of testing pupil's achievement in that subject and has worked out a form of test which will be described more fully in the subsequent paragraphs. We are interested in two main things in our testing, viz., we desire to see how well our pupils have learned the materials which we have given them, and secondly we wish to know what are the difficulties of each pupil and to obtain some data which will aid us in pointing out to him what he must do in order that he may improve himself.

It is unnecessary to point out that individuals differ in their ability to master a given subject and that they differ quite as much in their ability to comprehend sections within the subject. It is unnecessary to point out this fact, because most of us are familiar with it already and are anxious for some clew that will show us how a system may be devised that will not only give each child an opportunity to work up to his maximum ability but will enforce the opportunity by providing an incentive to him to so expend himself. But how can this be done when all are required to take the same work at the same time and in the same way as though, "all men are created equal"? The writer does not pre-

tend to answer this question but the method described below has proved helpful in at least a partial solution and for that much we can be thankful.

There are in general two types of problems with which pupils come in contact, viz., mechanical problems and verbal problems. The solutions of these two types of problems are very different both from the standpoint of difficulty and other matters involved. Nearly every teacher of mathematics knows how pupils dread verbal problems and how they usually have just cause for their dread. If then we are to devise a test in mathematics which will tell us what we need to know about our pupils we must have both mechanical and verbal problems in it.

However, upon examination we find that these types differ more in degree than in kind for a verbal problem becomes a mechanical problem when it is once translated from words to mathematical symbols. This being the case we are led to believe that translation is the difficult element in the verbal problem and it is therefore the element which we need most to test. In our tests, then, we should give many verbal problems but require the pupils merely to translate them to mathematical symbols and equations but not solve them completely.

The solution of mechanical problems involves many factors, more than it is possible to test in any single test, all of which contribute to the difficulty of scoring. We try to control some of these difficulties by grouping all the problems involving a particular operation together—for instance—in the test given below the two problems involving simple subtraction of a term from both sides of the equation are together. By such grouping we are able to make a better diagnosis of the particular type of difficulty which the pupil experiences.

There are two general types of mistakes which pupils make in examinations and a check for which must be provided in the tests. They are, mistakes which are constant, due to lack of knowledge or to having learned a wrong method of procedure; and secondly variable mistakes due to carelessness and the like. When a pupil comes to us for help, we must be able to tell him which type of mistake causes his difficulty. If, for instance, the pupil has a true conception of the algebraic method and makes a correct solu-

tion except that he carelessly makes a mistake in adding numbers, we will do him more good by providing him drill in the addition combinations than by having him solve addition problems in algebra—and particularly is this true if he makes a mistake habitually with certain combinations. Since individuals make different types of mistakes, they must have different types of remedial or drill work.

In order to provide this information we usually provide two problems involving the same kind of process. If the pupil makes the same type of mistake in both problems the chances are good that the mistake is characteristic. If, on the other hand, he misses one and solves the other correctly, he is either careless in or uncertain of his method. The difficulty being located it becomes an easy matter to find a remedy.

There are many factors which enter into the giving of tests and for which allowance is not always made by teachers. For instance, in our testing experience we have found that we obtain a greater percentage of error when we write our examinations on the blackboard and have the pupils solve them than when we provide each pupil with a copy of the examination. Also that better results are obtained when pupils are permitted to solve their problems on the examination paper than when they solve them on other paper and copy them on the examination paper. Not only has it proved better to provide examination papers for these reasons but it also makes it possible to standardize our scoring procedure and make the papers more uniform. Therefore we hectograph all examinations given to our classes and provide each pupil with a clear copy.

Any section of a term's work can be used for testing purposes. The tests may be made to fit the course as it has been taught to a given class and to emphasize any or all the principles which the class has covered. This is an advantage over the standardized test which must be made to fit general conditions. The following is a typical test which was given to a class of ninth grade students at the University High School during the first six weeks of their work last semester:

GENERAL MATHEMATICS

Page 1.

Name .... Oct. 12, 1922.

Solve the following equations for the value of the unknown terms:

$a + 9 = 29$	$3x - 8 = 7$
$a =$	$x =$
$m + 35 = 47$	$5y - 2.4 = 26$
$m =$	$y =$
$5x + 7 = 27$	$9x - 5x + 6.2 = 146$
$x =$	$x =$
$2y + \frac{14}{3} = 8\frac{2}{3}$	$7x + 3 = 4x + 12$
$y =$	$x =$
$6x + \frac{1}{2} = 30\frac{1}{2}$	$11x - 6 = x + 4$
$x =$	$x =$
$12x + .7 = 24.7$	$\frac{x}{2} - \frac{x}{4} = 3$
$x =$	$x =$
$2.5r + 4 = 129$	$2b + {}^{b} = 22$
$r =$	$b =$

GENERAL MATHEMATICS

Page 2.

Name .... Oct. 12, 1922.

1. Two numbers differ by 7. The smaller is *s*. Express the larger.

2. If $\frac{4}{5}$ of a number is decreased by 6, the result is 10. Find the number.

3. Find two consecutive odd numbers whose sum is 204.

4. Find the side of an equilateral triangle if the perimeter is thirty-six inches.

5. The length of a field is three times its width and the distance around the field is 200 rd. If the field is rectangular, what are the dimensions?

6. A and B own a house worth \$16,100 and A has invested twice as much capital as B. How much has each invested?

This test was scored readily because of the fact that we provided a place for the pupil to put the answer to each problem. A score card was made which had the correct answer placed so that it corresponded to the proper places on the test sheet. To score the test for achievement it was necessary to place the score card on the test sheet and check all answers which were unlike those on the score card. This can be done very rapidly and the students can do it as well as the teachers.

Such a score shows only achievement and taken alone it is practically worthless from the standpoint of diagnosis. Hence we have devised a chart which shows the results of the test both for the individual pupil and for the class for the purpose of diagnosis. There are two kinds of diagnosis with which we are concerned, they are; class diagnosis, and individual diagnosis. We are able to get valuable data concerning each of these from our chart.

Name	Equation and simple subtraction 2	Same with Coeff 1	Same with Com. Frac. 2	Same—Dec. Frac. 2	Equation and Simple Addition 2	Double Transposing 3	Equation—Com. Frac. 2	Verbal 6	Grade	Class Standing
L. M.						1	2	5	60	12
G. S.			2						90	4
J. S.						1		6	65	11
B. E.									100	1
C. B.	2	1	2	2	2	2	2	6	5	17
No. Problems to Solve	34	17	34	34	34	51	34	102		
No. Correct	31	16	25	27	26	43	16	56		
Per Cent Accuracy	91	94	73	80	80	84	47	55		

The chart has been abbreviated for the sake of space.

By totaling the vertical columns we find the total number of mistakes which the class made in any type of problem. If we compare this with the possible chances for mistakes we have a rough estimate of the efficiency of the class on this type problem. This we call class diagnosis for it tells a teacher wherein her instruction has been good or faulty. Now if we examine the horizontal rows after each pupil's name, we will easily discover which type or types of problems are causing his difficulty. For example: Column three gives the record for the problems involving fractions. Nine mistakes were made out of the thirty-four

chances making an efficiency of 73%. Again, in column seven we find another type of problem but involving fractions as before. Here we get an efficiency of 47%. Column eight shows in the same way that the class is only 55% efficient in the translation of verbal problems. Here then the teacher has information regarding the difficulties which her class is experiencing. She knows that she must emphasize fractions and the solution of verbal problems during the remedial period.

The chart shows that B. E. understands the work covered by this examination and that she is accurate in the fundamentals of arithmetic. B. E. need not take any remedial work and may work on in advance of the class and try her skill on something new. J. S. made one mistake due to a sign in the type of problem charted in column six. Since she worked the other problems correctly, this was probably accidental. However, she was unable to translate any of the verbal problems into the proper symbols. J. S. therefore must have drill in translation and need not concern herself very much with the solution of mechanical problems. C. B. was absent a great deal during this period and her chart shows that she knows practically nothing about it. She will require a great deal of attention during the remedial period.

The chart tells us only the kinds of problems missed by each pupil. It does not tell us the nature of the mistake. Since this is very apt to be different for each pupil, we get this information by indicating the mistake on the examination papers and returning them to the pupils. Each pupil makes a list of his mistakes. Then we are ready for our remedial work.

After giving such a test and indicating to each pupil just where he stands and as nearly as we can just why he stands there, we try to concentrate each pupil's efforts for a period of a few days to the end that these mistakes will be eliminated. The procedure during this period of remedial work is as follows: wherever we find a type of problem in which the class is not efficient, we reinstruct the entire group on this type; but if the class is proficient in a given type of problem while certain individual pupils are having difficulty with it, we take up the problem with these individuals during the supervised study period. By this method we minimize the boredom of having to work on something with which one is already sufficiently familiar.

A check test devised as nearly as possible like the first test and covering the same types of problems is given after this period of remedial instruction. It is charted and treated in the same way as the first test and gains or losses are noted for each pupil. This test gives a double check on all pupils who had apparently mastered the work the first time as well as showing what progress the individual pupils have made in the clearing-up of their deficiencies.

The following chart shows the results of two days' remedial work based on the diagnosis made from the chart given above and tested on the third day:

Name	Equation and simple subtraction 2	Same with coefficients 1	Same with Common Fraction 2	Same decimal fraction 2	Equation and simple addition 2	Double transposing 3	Equation—common fraction 2	Verbal 6	Grade	Class standing	Dept. Standing	Gain or loss
L. M.								2	90	7	15	+30
G. S.									100	1	1	+10
J. S.			2				1		85	10	22	+15
B. E.									100	1	1	0
C. B.				1		1	2	5	55	17	46	+50
No. Problems to solve	34	17	34	34	34	51	34	102				
No. Correctly solved	34	17	31	29	31	47	24	74				
Per. cent accuracy	100	100	91	86	91	92	70	73				

Column three has improved from seventy-three percent accuracy to ninety-one percent. Column seven has improved from forty-seven percent accuracy to seventy percent. Column eight shows improvement from fifty-five percent accuracy to seventy-three percent. Every column in the chart shows improvement.

L. M. has eliminated all mistakes except those in verbal problems and he shows great improvement in his work with them. G. S. has eliminated the error pointed out in the former chart. J. S. has a new type of mistake showing up, has made remarkable improvement in the solution of verbal problems. B. E. made no mistakes, as before. C. B. shows very great improvement considering the exceedingly poor showing made on the first test.

Conclusions

Teachers can devise tests which will measure the work which their pupils can do and at the same time furnish considerable information which will aid in the diagnosis of difficulties in the work.

Verbal problems and non-verbal problems differ only in the matter of translating the verbal into mathematical symbols. Both types must be tested in order to get a true measure of the pupil's ability.

Tests should show the teacher wherein the class is deficient and also point out to her each individuals' deficiencies.

Students make mistakes in their work from two general types of causes, first because they may lack the proper knowledge or skill (these may be called *constant*) and second, because they are careless or rushed, excited, fatigued, etc. (these may be called *variable.*)

When teachers devise tests that give them the necessary information concerning all these matters, there is a strong possibility that we may be able to get the pupils to work up to their capacities.

RECENT SYMBOLISMS FOR DECIMAL FRACTIONS

By FLORIAN CAJORI
University of California, Berkley, Cal.

Historians of mathematics have been so engrossed with the effort to reach an agreement on the question, who is entitled to the distinction of having first used the point or the comma for the specific purpose of separating the integral from the decimal part of a number, that they have failed to give attention to the decimal notations that have been used in more recent times. It will astonish many to learn that during the last 35 years over half a dozen different notations have been used in print.

For centuries there has been sharp competition between the point and the comma without either being able everywhere to dislodge the other. In 1616 Kepler proposed the comma; in 1617 John Napier suggested the "point or comma". During the seventeenth century the two symbols were in conflict with older notations, such as had been proposed by Vieta, Stevin, Beyer, Oughtred, Briggs and others. The older notations were clumsy to write and difficult to print, hence they succumbed in the struggle.

In the eighteenth century, trials of strength between the comma and the dot as the separatrix were complicated by the fact that Leibniz had proposed the dot as the symbol of multiplication, a proposal which was championed by the German textbook writer Christian Wolf and which met with favorable reception throughout the Continent. As a symbol for multiplication, the dot was seldom used in England during the eighteenth century, Oughtred's $\times$ being generally preferred. For this reason, the dot as a separatrix enjoyed an advantage in England during the eighteenth century which it did not enjoy on the Continent. Of 15 British books of that period, which we choose at random, 9 used the dot and 6 the comma. In the nineteenth century hardly any British authors employed the comma as separatrix.

In Germany, France and Spain the comma, during the eighteenth century, had the lead over the dot, as a separatrix.

During that century the most determined continental stand in favor of the dot was made in Belgium[1] and Italy[2].

But in recent years the comma has finally won out in both countries.

In the nineteenth century the dot became in England the favorite separatrix symbol. When the brilliant but erratic Rudolph Churchill critically spoke of the "damned little dots", he paid scant respect to what was dear to British mathematicians. In that century the dot came to serve in England in the double capacity, as the decimal symbol and as a symbol for multiplication.

Nor did these two dots introduce confusion, because (if we may use a situation suggested by Shakespeare) the symbols were placed in the Romeo and Juliet positions, the Juliet dot stood on high, above Romeo's reach, her joy reduced to a decimal over his departure, while Romeo below had his griefs *multiplied* and was "a thousand *times* the worse" for want of her light. Thus, $2{\cdot}5$ means $2\frac{5}{10}$, while 2.5 equals 10. It is difficult to bring about a general agreement of this kind, but it was achieved in Great Britain in the course of a little over half a century.

Charles Hutton[3] said in 1795:

"I place the point near the upper part of the figures, as was done also by Newton, a method which prevents the separatrix from being confounded with mere marks of punctuation." In Horsley's edition of Newton's *Arithmetica Universalis,* 1799, one finds in a few places the decimal notation "35'72"; it is here not the point but the comma that is placed on high. Probably as early as the time of Hutton the expression "decimal point" had come to be the synonym for "separatrix" and was used even when the symbol was not a point. In most places in Horsley's and Castillon's editions of Newton's works the *comma* 2,5 is used,

v

[1] Désiré André, *Des Notations Mathématiques, Paris, 1909 p. 19, 20.*

[2] Among eighteenth century writers in Italy using the dot are Paulino A. S. Josepho Lucensi who in his *Institutiones Analyticae,* Rome, 1738, uses

v

it in connection with on older symbolism, "3.05007"; G. M. della Torre, *Istituzioni arimmetiche,* Padua, 1768; Odoardo Gherli, *Elementi delle matematiche pure,* Modena, Tomo 1, 1770; Peter Ferroni, *Magnitudinum exponentialium logarithmorum et trigonometriae sublimis theoria,* Florence, 1782;F. A. Tortorella, *Aritmetica degl' idioti,* Naples, 1794.

[3] Ch. Hutton, *Mathematical and Philosophical Dictionary,* London, 1795, Art. "Decimal Fractions."

only in rare instances the *point* 2.5 The sign 2·5 was used in England by H. Clarke[1] as early as 1777. After the time of Hutton the 2·5 symbolism was adopted by Peter Barlow (1814) and James Mitchell (1823) in their mathematical dictionaries. Augustus De Morgan states in his *Arithmetic*[2]: "The student is recommended always to write the decimal point in a line with the top of the figures, or in the middle, as is done here, and never at the bottom. The reason is that it is usual in the higher branches of mathematics to use a point placed between two numbers or letters which are multiplied together." A similar statement is made in 1852 by T. P. Kirkman[3]. Finally, the use of this notation in Todhunter's texts secured its general adoption in Great Britain.

Newton's idea of extending the usefulness of the *comma* or *point* by assigning it different vertical positions, occasionally met with favor among writers in other countries. Thus, we find the decimal comma placed in an elevated position 2'5 by Louis Bertrand[4] in Geneva, Switzerland, by Daniel Adams[5], in New Hampshire, who in 1827 used 2'5, and by a number of writers of the nineteenth and twentieth centuries in Spain and Spanish speaking countries[6].

Books in the Spanish language sometimes contain the elevated comma inverted, thus 2‘5. Other writers use an inverted wedge-shaped comma[7], in a lower position, thus 2,5. In Scandinavia and Denmark the dot and the comma have had a very close race, the comma being now in the lead. The practice is also widely prevalent, in those countries, of printing the decimal part of a

---

[1] H. Clarke, *Rationale of Circulating numbers*, London, 1777.

[2] A. De Morgan *Elements of Arithmetic*, 4th edition, London, 1840, p. 72

[3] T. P. Kirkman, *First Mnemonical Lessons in Geometry, Algebra and Trigonometry*, London, 1852, p. 6.

[4] L. Bertrand, *Developpment nouveaux de la partie elementaire des mathématiques*, Tome I, Geneva, 1778, p. 7.

[5] Daniel Adams, *Arithmetic*, Keene, N. H., p. 132.

[6] See, for instance, Don Gabriel Ciscar, *Curso de estudios elementales de Marina*, Mexico, 1825, who writes 9'75; D. Federico Villareal, *Calculo Binomal*, P. I., Lima, (Peru), 1898), p. 416, who writes 0'2; D. J. Cortazár, *Trartado de Aritmética*, 42. edición, Madrid, 1904, who writes 5'3.

[7] As in A. F. Vallin, *Aritmética para los miños*, 41. edición, Madrid, 1889, p. 66.

number in smaller type than the integral part.[1] Thus we frequently find there the notations $2,_5$ and $2._5$. To sum up, in books printed within 35 years we have found the decimal notations 2.5, 2·5, 2,5, 2'5, 2‘5, $2_\wedge 5$, $2,_5$, $2._5$.

In the United States the decimal point has always had the lead over the comma, but the latter part of the 18th century and the first half of the 19th century the comma in the position of 2,5 was used quite extensively. During 1825-1850 it was the influence of French texts which favored the comma. We have seen that Daniel Adams used 2‘5. Since about 1850 the dot has been used almost exclusively. Several times the English elevated dot was used in books printed in the United States. The notation 2·5 is found in American editions of Hutton's Course of Mathematics that appeared in the interval 1812-1831, in Samuel Webber's Mathematics (Boston 1801), in William Griev's Mechanics Calculator, from the 5th Glasgow edition, Philadelphia, (1842), in Thomas Sherwin's Common School Algebra, Boston, 1867, (first edition 1845), in George R. Perkins' Practical Arithmetic, New York, 1852. Says Sherwin: "To distinguish the sign of Multiplication from the period used as a decimal point, the latter is elevated by inverting the type, while the former is larger and placed down even with the lower extremities of the figures or letters between which it stands."

It is difficult to assign definitely the reason why the notation 2·5 failed of general adoption in the United States. Perhaps it was due to mere chance. Men of influence, such as Benjamin Peirce, Elias Lôomis, Charles Davies, Edward Olney, did not happen to become interested in this detail. America had no one of the influence of De Morgan and Todhunter in England, to force the issue in favor of 2·5. As a result 2.5 had for a while in America a double meaning, namely 2 5/10 and 2 times 5. As long as the dot was seldom used to express multiplication, no great inconvenience resulted, but about 1880 the need of a distinction clearly arose. The decimal notation, being at that time thoroughly established in this country as 2.5, the dot for multi-

[1] Gustaf Haglund, *Samlying of Öfningsexempel till Lärabok i Algebra.* Fjerde Upplagan, Stockholm, 1884, p. 19; *Öfverigt af Kongl. Vetenskaps-Akademiens Förhandlingar.* Vol. 59, 1902, Stockholm, 1902, 1903, p. 183, 329; *Oversigt over det Kongelige Danske Videnskabernes Selskabs, Fordhandlinger,* 1915, Kobenhavn, 1915, p. 33, 35, 481, 493 545.

plication was elevated to a central position. Thus 2·5 means with us 2 times 5.

Comparing our present practice with the British the situation is this: We write the decimal point low, they write it high; we place the multiplication dot half way up, they place it low.

NEWS AND NOTES

The Association of Teachers of Mathematics in New England held the mid-Winter meeting at Hartford Public High School, Hartford, Connecticut, Saturday, March 10, 1923. The program included: (1) Live Problem Material for Algebra, by Dwight S. Davis, Athol High School; (2) Practical Teaching in Mathematics, by Professor Joshua I. Tracey, Yale University; (3) College Board Mathematics Examinations from Makers to Readers, by Harry B. Marsh, Technical High School, Springfield; (4) Varieties of Space, by Professor Emily N. Martin, Mount Holyoke College; (5) Maps, by Professor William R. Ransom, Tufts College.

The Council for 1923 consists of A. Harry Wheeler, President, North High School, Worcester; Professor Lennie P. Copeland, Vice-President, Wellesley College; Harry D. Gaylord, Secretary, Browne and Nichols School, Cambridge, Address, 448 Audubon Road, Boston; Harold B. Garland, Treasurer, High School of Commerce, Boston; William L. Vosburgh, Boston Normal School; Miss Gertrude E. Preston, Dana Hall, Wellesley, Mass.; Professor J. W. Young, Dartmouth College; Charles H. Mergendahl, Newton Classical High School; Harry C. Barber, English High School, Boston; Miss Olive A. Kee, Boston Normal School.

The Southeastern Section of the Mathematical Association of America held its regular meeting this year at Agnes Scott College, Decatur, Ga., on Saturday March 10.

Professor David Eugene Smith, of Columbia University, was the principal speaker. It will be of interest to know that Professor Smith has in press now a two-volume History of Mathematics. Professor H. E. Slaught, of the University of Chicago, and well known to all teachers of mathematics through the Slaught-Lennes text-books, also addressed the Association.

A special dinner in honor of Professors Smith and Slaught was held at the Agnes Scott Alumnae House.

The program included: (1) Address by Professor David Eugene Smith, Columbia University, "The Teaching of the

History of Mathematics in College"; (2) Professor J. B. Coleman, University of South Carolina, "Correlated Mathematics"; (3) Professor H. E. Slaught, University of Chicago, "The Reasons Why I Am a Devotee of Mathematics"; (4) Professor A. W. Hobbs, University of North Carolina, "Unified Mathematics for Freshmen". Professor W. W. Rankin, Jr., of Agnes Scott College, arranged the program.

The issue of School and Society for Jan. 20 contains an article entitled "A Mathematician on the Present Status of the Formal Discipline Controversy," by Professor N. J. Lennes, of the State University of Montana. This article should be read by every teacher of mathematics and by others. In conclusion the writer says:

"Psychological criticism and experiments have left us largely where we were two generations ago in regard to the fundamental question of education which we are considering—in this field we are as free to pass common sense judgments as we ever were." "Finally it must be said that the last two decades have witnessed a change in the attitude of the psychologists. The question at issue now seems to be not whether transfer actually takes place but how it takes place."

ALFRED DAVIS

The Twentieth Annual Meeting of the Kansas Association of Mathematics Teachers was held at Topeka, Kansas, January 20, 1923.

The program consisted of: 1. The Development of the Junior High School Movement in Kansas and Its Effect on the Efficiency of Instruction in Mathematics, by Professor Theodore Lindquist, Kansas State Normal School. 2. Should the State Board of Education Recognize the Existence of the Junior High School System in the State? General Discussion. 3. The National Committee's Report on the Reorganization of Secondary School Mathematics, by Miss Eula A. Weeks, Cleveland High School, St. Louis, Mo. 4. Content of the Mathematics Course in the Junior High School. (a) Arithmetic, by Principal Evan E. Evans, Neodesha High School. (b) Algebra, by Miss Alice Bell, Horace Mann Junior High School, Wichita. (c) Geometry, by Miss Amy Irene Moore, Leavenworth High

School. 5. The Unification of the Junior and Senior High School Courses in Mathematics, by Miss Bernice Boyles, Topeka High School. 6. Some Junior High School Mathematics Textbooks. Discussion led by Miss Irma Long, Lawrence Junior High School.

The officers are: President, W. H. Garrett, Baker University; Vice-President, Garnett M. Everly, Emporia High School; Secretary-Treasurer, Edna E. Austin, Topeka High School.

The Ninth Regular Meeting of the Kansas Section of the Mathematical Association of America was held at Topeka, Kansas, January 20, 1923, jointly with a meeting of the Kansas Association of Mathematics Teachers. The program included: 1. The development of the Junior High School Movement in Kansas and Its Effect on the Efficiency of Mathematics Instruction in the Seventh, Eighth and Ninth Grades, by Professor Theodore Lindquist, Kansas State Normal School. 2. Should the State Board of Education Recognize the Existence of the Junior High School System in the State, General Discussion. 3. The National Committee's Report on the Reorganization of Secondary School Mathematics, by Miss Eula A. Weeks, Cleveland High School, St. Louis, Mo. 4. Some Peculiar and Limiting Functions and their Graphs, Professor Guy W. Smith, University of Kansas. 5. The Teaching of Unified Mathematics, Professor Pius Pretz, St. Benedict's College. 6. The Area of a Cone Having an Elliptical Base, Miss Thurza Mossman, Kansas State Agricultural College. 7. A New Method of Determining Sufficient Conditions for Real Roots of Equations, Miss Wealthy Babcock, University of Kansas. 8. A Map of Sinh Z, Professor T. B. Henry, Highland College.

The officers of the Kansas Section are: A. E. White, Chairman; Theodore Lindquist, Vice-Chairman; W. G. Mitchell, Secretary.

DISCUSSION

To the Editor of The Mathematics Teacher—In your issue of November, 1922, Professor Bessie I. Miller includes in a "reading list for browsing" *Fundamental Concepts of Modern Mathematics* by Richardson and Landis with the comment: "This is one of the disappointments of this generation. Its mathematics cannot be trusted."

This is so sweeping a condemnation that we feel Miss Miller ought to make some attempt to justify it. She will not, we contend, be able to show misrepresentation i nthe views we ascribe to those mathematicians from whom we have been obliged to differ. As for the doctrines we adhere to and ourselves put forward, we do not ask that these be taken on trust. Reason and not blind adhesion to authority is, in our opinion, the touchstone in mathematics.

Our work is controversial in its nature; and is designed, not for those who seek to follow authority, but for those who prefer to face plainly the doubts and difficulties of debated questions and reach decisions of their own. Any stimulus we may give toward leading a student to decide such matters in the light of his own intellect is, we hold, a benefit, whether he reaches the same conclusions as ourselves, or attains other that are quite different.

For those who limit their studies to mathematical symbolism, and care not to inquire into the realities of nature which these symbols may represent, our investigations can have no interest. And we have never entertained the hope of reaching an agreement with the disciples of a certain school of philosophy which, notwithstanding the vogue enjoyed by Bertrand Russell and other advocates of its doctrines, is regarded as false by many thinkers.

Any profitable discussion of the inquiries undertaken in our work would have to take as basis an examination into the realities underlying the symbolic expressions in which mathematics is formulated. But no attempt has so far been made to controvert our theories from such a viewpoint.

To relegate the topics of our work to some indefinite region